FORM, SHAPE AND SPACE: AN EXHIBITION OF TILINGS AND POLYHEDRA

10 OCTOBER, 2007 – 16 MAY, 2008

TEACHERS' BOOKLET

University of Leeds International Textiles Archive (ULITA)
St. Wilfred's Chapel, Maurice Keyworth Building
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Form, Shape and Space:
An Exhibition of Tilings and Polyhedra

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1. Introduction to the Exhibition

Symmetry is a simple concept that has tremendous meaning. It is possibly the most significant and elegant connection that transcends the boundaries between art, science and mathematics. Symmetry surrounds us, both in the natural world and in the world conceived by humans. Patterns and proportional relationships create a visual language expressing order and generating appealing, fascinating compositions. In the natural world symmetrical patterns can be found on every conceivable scale. Microscopic organisms, viruses and crystal structures, all exhibit the mathematical regularities of symmetry.

One of the most practical applications of symmetry, drawing on concepts originating in the study of molecular crystal structures, is in the analysis and construction of regularly repeating patterns and tilings. Following certain geometrical rules, a wealth of patterns can be created, using the economical power of symmetry. This notion of modularity embraces the concept of 'minimum inventory and maximum diversity', which pervades the disciplines of science, art and design, as well, as being found throughout the natural and human-made world. From a few basic elements (or modules) an extensive number of solutions (in this case patterns) may unfold, offering great innovative potential in design, the decorative arts and architecture.

In three-dimensions, a polyhedron (also known as a mathematical solid) consists of polygonal faces, with these faces meeting at edges, and edges joining at vertices. Within each Platonic solid the faces are thus identical in size and shape, and the same number of faces meets at each vertex. There is a set of polyhedra, known as the Platonic or regular solids, which are composed of combinations of one specific type of regular polygon. The five highly symmetrical Platonic solids are as follows: the tetrahedron (four faces), the octahedron (eight faces), the cube or hexahedron (six faces), the dodecahedron (twelve faces) and the icosahedron (twenty faces).

The symmetry characteristics that govern the properties of regularly repeating patterns and tilings, are also of importance to three-dimensional solids. This project considers the links between symmetry in two- and three-dimensions and examines how these characteristics govern the application of repeating patterns to the surface Platonic solids. An understanding of the symmetry in two- and three-dimensions can help provide a means by which patterns can be applied to repeat across the surface of polyhedra, avoiding gaps or overlaps, and ensuring precise registration.

Inspired by the geometric designs at the Alhambra Palace in Granada, Spain, this exhibition is comprised of a substantial collection of two-dimensional pattern designs and three-dimensional patterned polyhedra. The collection is displayed in pairs of cabinets with each pair named after an area of the Alhambra Palace.

Figure 1 *The icosahedron regularly tiled with pattern class p6*

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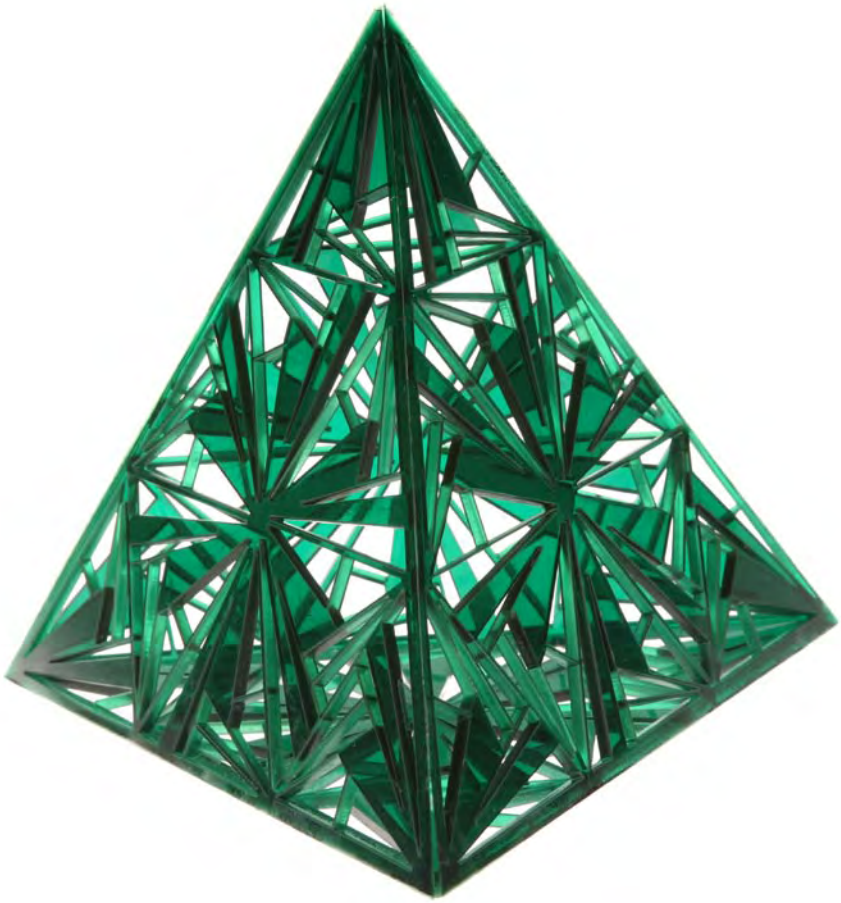


Figure 2 *The tetrahedron regularly tiled with pattern class p6*

Figure 3 *The dodecahedron regularly tiled with pattern class p6mm*

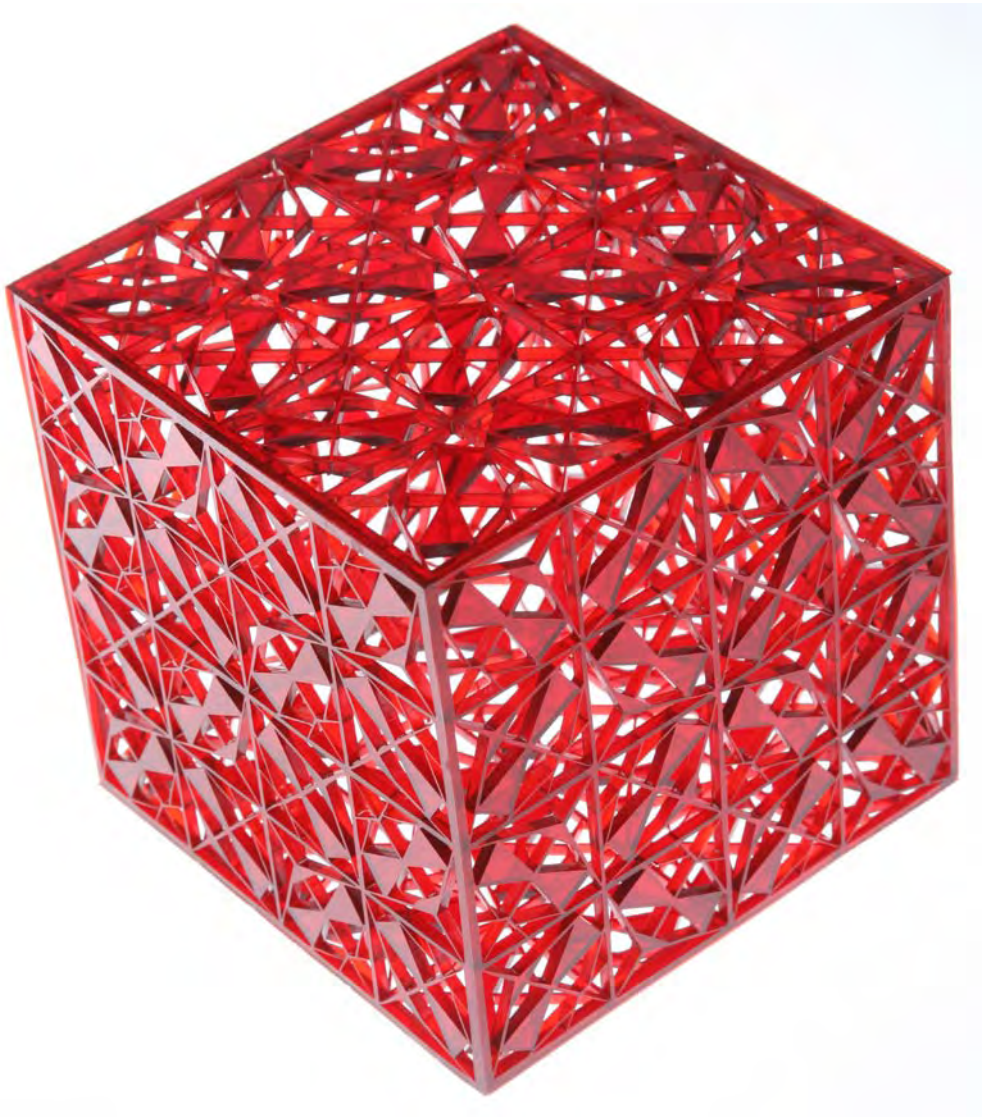


Figure 4 *The cube regularly tiled with pattern class p4*

Figure 5 *The octahedron regularly tiled with pattern class p3m1*

2. Islamic Art

Traditional Islamic art may appear unusual to those of us who are more familiar with post-Renaissance European art. Islamic art is dominated by lavish geometric decoration consisting of texture, colour, pattern and calligraphy. The key to its understanding it is to remember that the exquisite designs are not purely decorative – they represent a spiritual vision of the world – a reminder of the 'Unity of God'.

The word Islam means 'surrender' which may be generally interpreted as meaning submission to the will of God, known as Allah. The Islamic faith is based on the Islamic holy book, the *Qur'an* (sometimes spelt *Koran*). Followers of Islam believe the Qur'an to be the word of God, as revealed through the Archangel Gabriel to the Prophet Mohammed in the early seventh century. The Prophet was born in Arabia in around 571 AD and died in 632 AD. By the early eighth century Islam had spread by military conquest westward as far as Spain and eastward to Samarkand and the Indus Valley. Islam continued to expand, into Turkey and deeper into the Indian subcontinent, into north-western China and South-east Asia. The followers of Islam are known as *Muslims*, and today Islam has become the second most popular religion in the world.

Islamic art avoids depicting human and animal forms. This contrasts with European religious art where the central theme is the human image. There are several reasons for this. Firstly, the Qur'an condemns idolatry. Nothing should come between the Muslim at prayer and the invisible presence of God. Secondly, God cannot be represented visually. To do this is to limit him. Thirdly, man is made in the image of God and to imitate his form is tantamount to blasphemy. Certain Islamic texts (produced after the Qur'an forbid the depiction of living creatures and also the description of beings having a soul. Finally, Islamic art aims to express the order and harmony of the cosmos – divine unity. This concept cannot be expressed through an image which, by its nature, is constraining and finite. However, humans and animals do appear in Islamic secular arts, such as Persian miniatures. The three principle elements of Islamic art are calligraphy, arabesque and geometry.

Calligraphy is of great significance in Islamic culture. In various forms of Arabic script, calligraphy is the visual form of the word of God, the Qur'an. The geometric patterns of the letters are based on ancient laws that the crafts person follows. Through the careful use of a pair of compasses and a ruler, beautiful and harmonious compositions of design and colour can be achieved (see figure 6).

Arabesques are seemingly infinite, curling foliage or scrollwork which represent the essential rhythm of nature. An example of an arabesque found at the Alhambra Palace in Granada, Spain is shown in figure 7.

Figure 6 *Calligraphy found at the Alhambra Palace, Granada, Spain*

Figure 7 *An arabesque found at the Alhambra Palace, Granada, Spain*

Geometry is considered to be at the heart of nature. For this reason it is also at the heart of Islamic art. Islamic pattern designs use a finite number of geometric shapes which combine in many different ways. (More about this later.)

These three elements of calligraphy, arabesque and geometry are often combined within the decorative scheme on a single object. Traditionally, works of art and design are all made to the glory of Allah, whether the objects are for use in the mosque, which is the place for prayer, or in the home. The mosque is the centre of religious life and each mosque has its own unique design. Mosques are typically characterised by highly decorative tile work and painted walls with amazing richly coloured geometric patterns.

2.1 Geometry in Islamic art

As mentioned previously, many people consider geometric principles to be at the heart of nature. It can be seen for example in the symmetrical shapes of flowers, in the hexagonal pattern of bee hives and in the spiral patterns of a pine cone. These patterns depict the Islamic principles of *tawhid* (the unity of all things) and *mizan* (order and balance), which are the principal laws of creation in Islamic art.

Repetition and variation are important aspects of Islamic design. A series of tiles may consist of only one or two shapes but the patterns on the tiles may all be different. In other designs, a few different shapes may be combined to create a complex interlocking pattern (see figure 8).

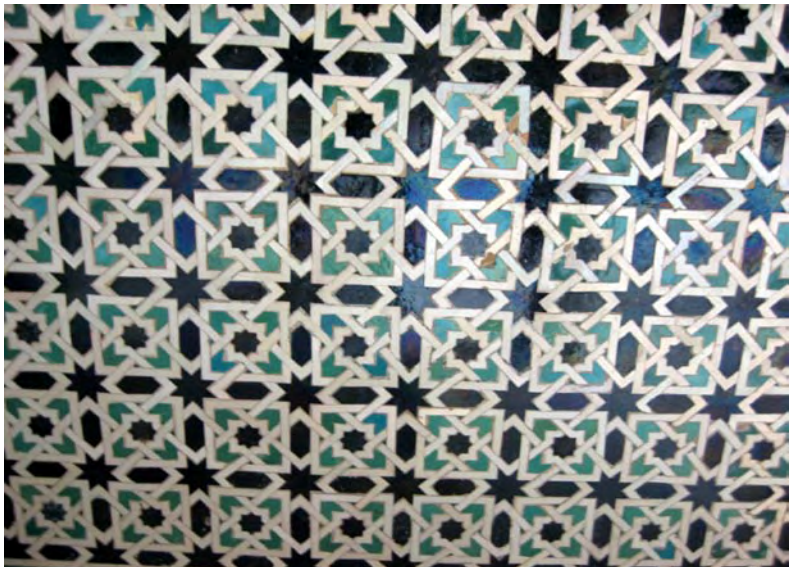


Figure 8 *A tiling pattern found at the Alcazar, Seville, Spain*

Symmetry plays an important part in Islamic patterns. There may be a single line of reflective symmetry, from top to bottom or across the middle, or there may be three or four lines of symmetry. Straight (translation) and turning (rotational) movements are also used. Often reflective symmetry, translation and rotational transformations are found in the same design. Symmetry and repetition impart unity to the more complex designs.

In many Islamic patterns, different elements seem to dominate, depending on how the patterns are viewed. A simple example is shown in figure 9. Perhaps you see it as a six-pointed star surrounded by six rhombi, or as three rhombi surrounded by three six-pointed stars?

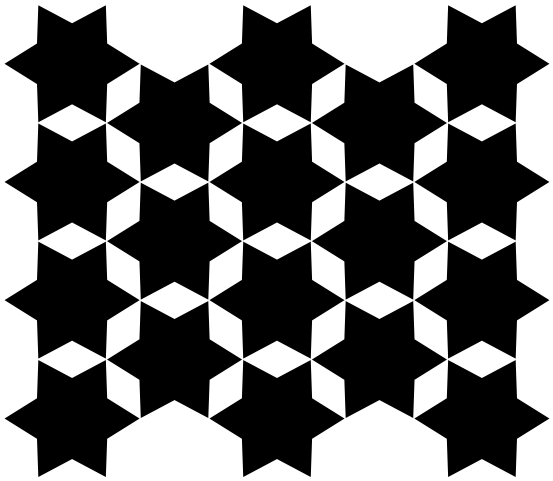


Figure 9 *Six-pointed star tiling*

2.2 Symbolism in Islamic art

Patterns are symbolic in Islamic culture. It is within the circle, the symbol of unity, that all patterns and polygons are developed. The circle has also been regarded as a symbol of eternity, without a beginning and without an end.

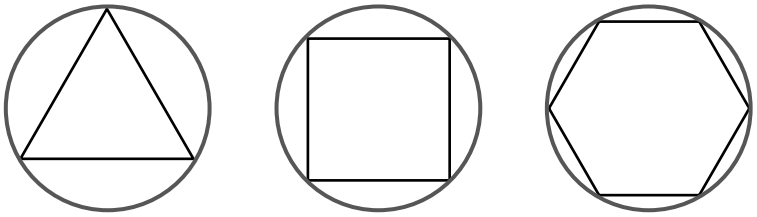


Figure 10 *Illustrations of a triangle, a square and a hexagon inscribed within circles*

From the circle we get the three fundamental shapes used in Islamic art – the equilateral triangle, the square and the hexagon (see figure 10).

- The equilateral triangle, the simplest regular polygon, is made with only three equal lines. Two lines cannot enclose a plane. Three are needed and so three is the beginning. By tradition the triangle is a symbol of the principle of harmony and human consciousness.
- The square is often taken as the symbol to represent the earth and its four corners symbolically represent the four directions – north, south, east and west or the four states of matter – water/liquid, earth/solid, air/gas and fire/ether.
- The hexagon represents heaven.

Another symbol which is often found in Islamic art is the star. The star symbolises equal distance in all directions from a central point. All stars whether they have 6, 8, 10, 12 etc points are created by dividing the circle into equal parts. The centre of the star is the centre of the circle from which it was created, and its points touch the circumference of the circle. The rays of the star spread out in all directions making the star an appropriate symbol for the spread of Islam.

3. The Alhambra Palace

The name of Alhambra is an abbreviation of the Arabic, *Qal'at al-Hamra* (the red fort), which describes the reddish colour of the walls of the palace. The Alhambra was built in southern Spain as a fortified town within the city of Granada. It is surrounded by a 2.2 kilometre wall with 22 towers and four main gates. The palace's known history began in the 11th century, but most of the buildings, towers and gardens for which the Alhambra is famous were built during the reign of Moorish rulers in the period 1238 – 1381 AD. Further construction work was undertaken after the Muslim rule in 1492. This additional work, however, was not in-keeping with the original style. After the 16th century the Alhambra fell into a state of decay and restoration only began after a visit by the American writer, Washington Irving, whose book *The Alhambra* (1832) aroused world wide interest in its architectural wonders. In 1984 the palace was included as part of the World Heritage List of UNESCO.

Mathematically the Alhambra is famous for its geometric decorations on the walls, ceilings and panels in the halls and courts of the Muslim palaces: the Mexuar, the Palacio de Comares and, the most celebrated, the Palacio de Leones. These richly coloured decorations stop the visitor in their tracks. How can such awesomely intricate designs be based simply on the regular division of a circle?

Figure 11 *Palacio de Leones (Courtyard of the Lions), the Alhambra Palace, Granada, Spain*

4. The Platonic Solids

The Platonic solids are regular polyhedra. "Polyhedra" is a Greek word meaning "many faces." There are five of these Platonic solids and they are characterised by the fact that each face is a regular polygon - a polygon with equal sides and equal angles.

Tetrahedron
4 equilateral triangular faces, 4 vertices and 6 edges.

Cube
6 square faces, 8 vertices and 12 edges.

Octahedron
8 equilateral triangular faces, 6 vertices, and 12 edges.

Dodecahedron
12 regular pentagonal faces, 20 vertices and 30 edges.

Icosahedron
20 equilateral triangular faces, 12 vertices and 30 edges.

Figure 12 *The Platonic Solids*

Why there are only exactly five Platonic solids? The key to it all is to remember that the interior angles of the polygons meeting at a vertex of a polyhedron must add up to less than 360 degrees. If they total 360 degrees a plane tessellation is created. If they total more than 360 degrees a vertex cannot exist. Also consider that there must be three or more polygons meeting at each vertex. Use these facts to consider all the possibilities for the number of faces meeting at a vertex of a regular polyhedron. Here are the possibilities:

- Triangles: The interior angle of an equilateral triangle is 60 degrees. Thus on a regular polyhedron, only 3, 4, or 5 triangles can meet a vertex. If there were more than 6 their angles would add up to more than 360 degrees. Consider the possibilities:
  3 triangles meet at each vertex. This gives rise to a tetrahedron.
  4 triangles meet at each vertex. This gives rise to an octahedron.
  5 triangles meet at each vertex. This gives rise to an icosahedron.

- Squares: Since the interior angle of a square is 90 degrees, only three squares can meet at a vertex. This gives rise to a hexahedron or cube.

- Pentagons: As in the case of cubes, the only possibility is that three pentagons meet at a vertex. This gives rise to a dodecahedron.

- Hexagons or regular polygons with more than six sides cannot form the faces of a regular polyhedron since their interior angles are at least 120 degrees.

Euclid and other Ancient Greeks philosophers (around 300 BC), in their love for geometry, called these five solids the atoms of the Universe. In the same way that we today believe that all matter is made of different arrangements of atoms, so the Ancient Greeks believed that all physical matter was made of the atoms of the Platonic solids. The Ancient Greeks also believed that all matter was mystically represented by their connection with earth, air, fire, water and ether.

Plato wrote of the four elements:

We must proceed to distribute the figures [the solids] we have just described between fire, earth, water, and air …

Let us assign the cube to earth, for it is the most immobile of the four bodies and most retentive of shape

The least mobile of the remaining figures (icosahedron) to water
The most mobile (tetrahedron) to fire
The intermediate (octahedron) to air

Significantly and rather surprisingly he associated the earth with the cube, with its six square faces. This lends support to the notion of the four-squaredness of the earth.

Plato also wrote.

There still remained a fifth construction, which the god used for embroidering the constellations on the whole heaven.

Plato's statement is vague, and he gives no further explanation. Later, Greek philosophers assigned the dodecahedron to the ether or heaven or the cosmos. The dodecahedron has 12 faces, and modern number symbolism associates 12 with the zodiac. This might be Plato's meaning when he writes of "embroidering the constellations" on the dodecahedron.

Similar to our modern understanding of the atom, which shows a nucleus surrounded by electrons in orbits creating spheres of energy, the Greeks felt that the Platonic solids had a spherical property. A Platonic solid inscribed within a sphere, alternately fits inside another Platonic solid, again fitting in another sphere and so on.

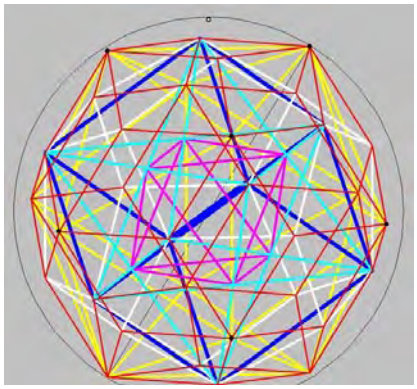


Figure 13 *The Platonic solids inside a rhombic triacontahedron, surrounded by a sphere*

The 5 nested Platonic Solids are illustrated inscribed inside a rhombic triacontahedron, surrounded by a sphere, in figure 13. The icosahedron in cream, the rhombic triacontahedron in red, the dodecahedron in white, the cube in blue, two interlocking tetrahedra in cyan, and the octahedron in magenta (for those of you viewing in colour). Only the 12 vertices of the icosahedron touch the sphere boundary.

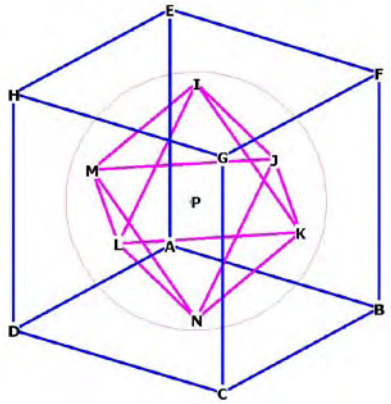


Figure 14 *An octahedron inside a sphere inside a cube*

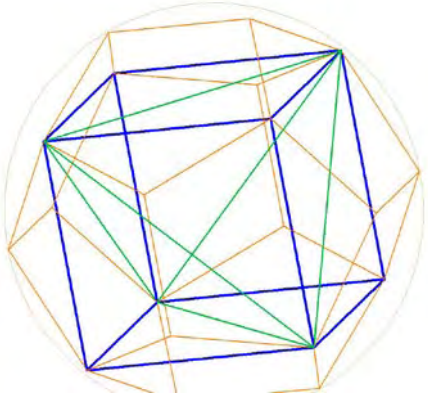


Figure 15 *A tetrahedron inside a cube inside a dodecahedron inside a sphere*

In the present day this might seem very geometrical, naive and far fetched. Yet these still proved to be very powerful ideas in later centuries.

As late as the 16th century, the astronomer Johannes Kepler applied a similar intuitive idea to attempt to explain planetary motion. Early in his life he had reasoned that the distances of the orbits, which he assumed were circular, were related to the Platonic solids in their proportions. This model is represented in the woodcut shown in figure 16, taken from his treatise *Mysterium Cosmographicum*.

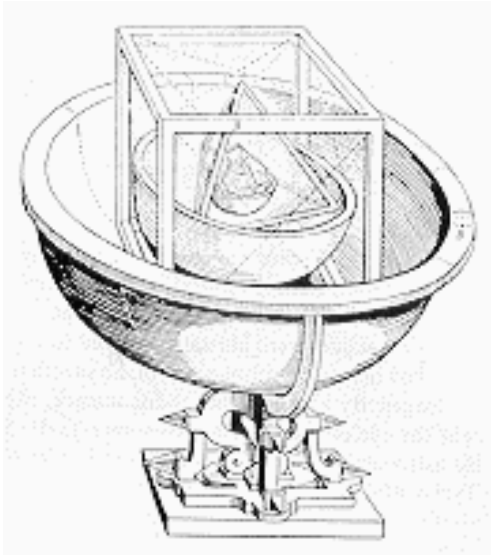


Figure 16 *Kepler's model of planetary motion*

Only later in life, after the death of his friend the great astronomer Tycho Brahe, did Kepler finally come to the conclusion that this model of planetary motion was flawed. Tycho had left him an enormous collection of astronomical observations and Kepler used this collection to conclude that planets moved around the sun in ellipses, not circles. It was this discovery that led Isaac Newton, less than a century later, to formulate his law of gravity – which governs planetary motion – and that ultimately gave us our modern conception of the universe.

The elegance and beauty of the Platonic solids has inspired artists throughout the centuries – not just mathematicians and scientists. For example, polyhedra are a predominant feature in M.C. Escher's 1948 engraving, *Stars*, shown in figure 17.

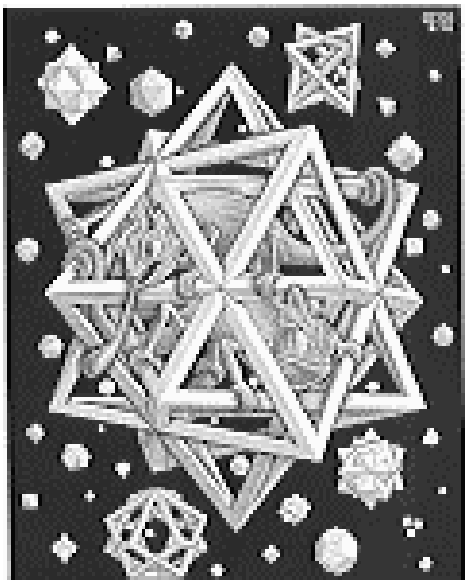


Figure 17 *M.C. Escher, Stars, 1948*

Escher so loved the Platonic solids he made a nested set of them and when he moved to a new design studio, he gave away most of his belongings but kept his much loved model (shown in figure 18).

Figure 18 *M.C. Escher and his polyhedron model*

More recently Paul Carter built a set of Platonic solids outdoors from tree branches, see figure 19.

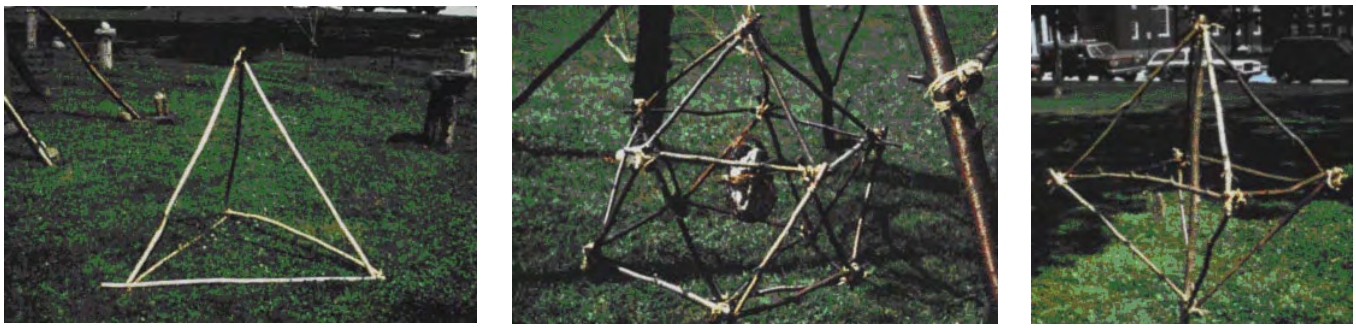


Figure 19 *Paul Carter's Platonic solids*

Summary of the Characteristics of the Platonic Solids

Platonic Solid	**Faces**	**Shape of Faces**	**Faces at each Vertex**	**Vertices**	**Edges**	**Dual (formed by connecting mid-points of faces)**
Tetrahedron	4	Equilateral triangle	3	4	6	Tetrahedron
Cube	6	Square	3	8	12	Octahedron
Octahedron	8	Equilateral triangle	4	6	12	Cube
Dodecahedron	12	Pentagon	3	20	30	Icosahedron
Icosahedron	20	Equilateral triangle	5	12	30	Dodecahedron

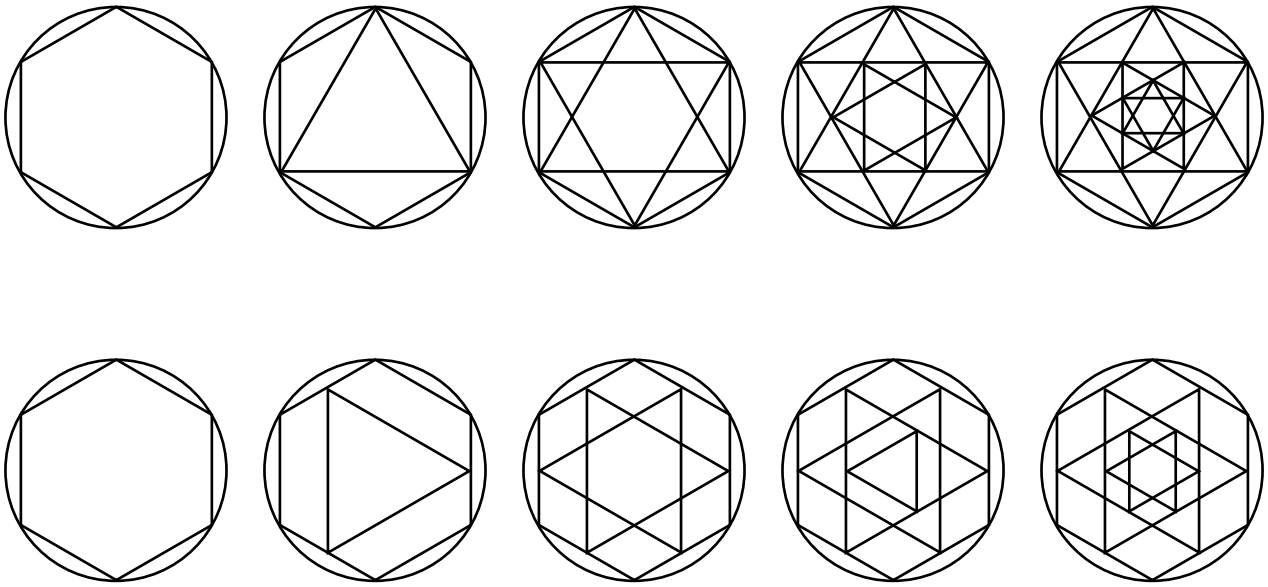
5. Activities

5.1 Making shapes: equilateral triangles, hexagons and six-pointed stars using a compass and ruler

Some of the most common shapes used in Islamic decorations are equilateral triangles, hexagons and stars. These can be made using a compass and a straight edge (usually a ruler).

Instructions for drawing equilateral triangles, hexagons and stars

- Draw a circle with a compass.
- Put the compass point anywhere on the circumference of the circle and then mark another point on the circumference.
- Move the point of the compass to this new point (pencil mark) and make another pencil mark on the circumference.
- Continue doing this around the circumference until there are six marks
- From these six points a series of hexagons, equilateral triangles and 6-pointed stars can be made.

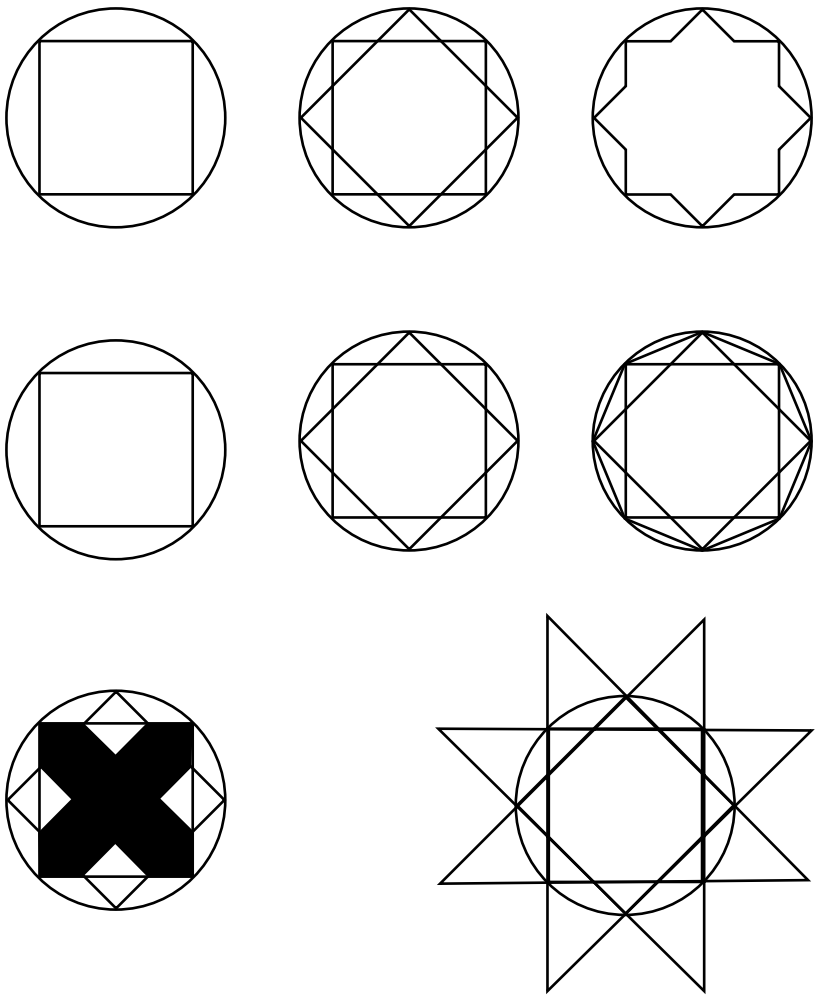


5.2 Making shapes: squares, octagons, crosses and eight–pointed stars using a compass and ruler

Some of the most common shapes used in Islamic decorations are squares, octagons and 8-pointed stars.

Instructions for drawing squares, octagons, stars and crosses

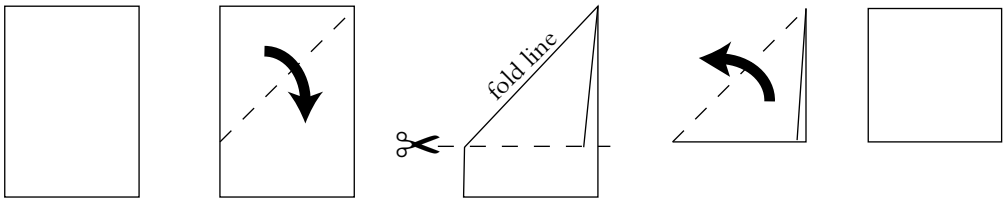
- Draw a circle with a compass.
- Draw a diameter with a ruler and pencil.
- Construct the perpendicular diameter through the centre.
- Join up the four points on the circumference and rub out the two diameters
- Construct the perpendicular bisectors of two adjacent sides of the square. Extend these lines to cut the circumference at four points. Join each of these new points to form a second square. An 8-pointed star is then made.
- From this 8-pointed star you can make a large octagon and a smaller one and then extend the sides of the squares to form a larger 8-pointed star. A cross can be made from the first 8-pointed star.



5.3 Making shapes: paper-folding polygons

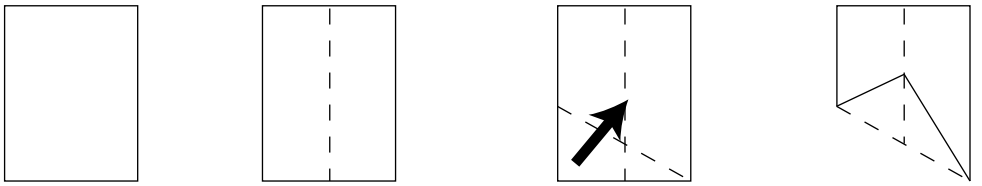
Always start with a piece of 'A' sized paper - this could be any 'A' size (it doesn't have to be A4), depending on how large a shape you want.

a) Square

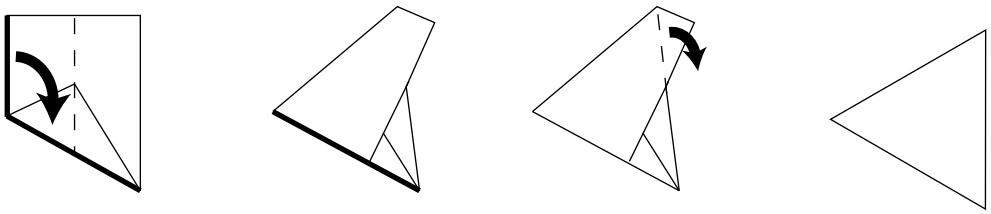


fold paper | cut strip off bottom | unfold paper

b) Equilateral triangle

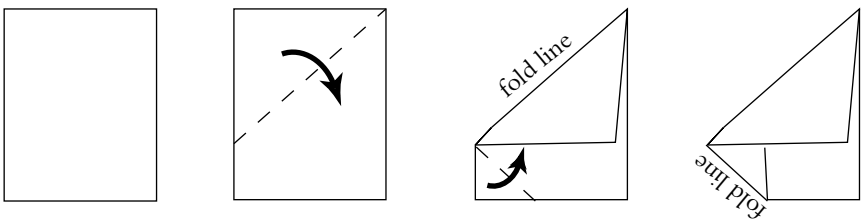


fold in half | fold corner to middle line



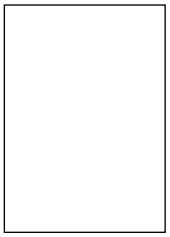
fold edge to edge | fold over triangle flap

c) Kite

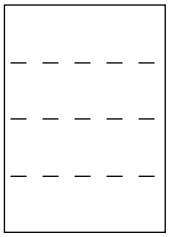


fold paper | fold bottom strip

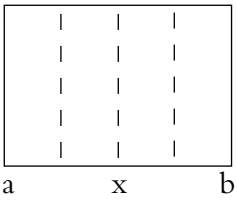
d) Hexagon



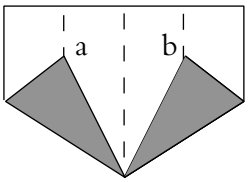
A size paper



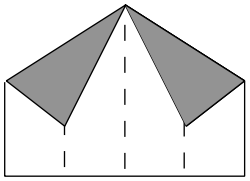
fold in half, then in quarters and unfold



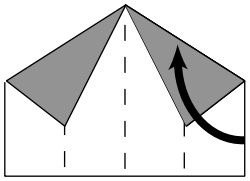
rotate paper



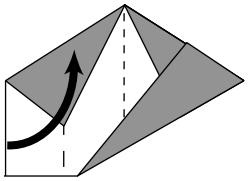
roll corners A and B along quarter fold lines to make lines AX and BX



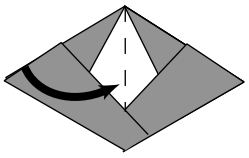
rotate shape



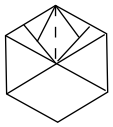
fold edge to edge



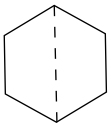
fold edge to edge



fold corners to centre using quarter fold lines

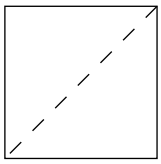


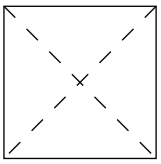
flip over



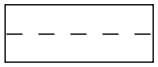
regular hexagon

e) Octagon

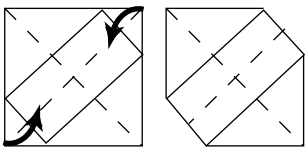




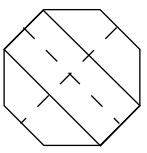
fold corner to corner and unfold



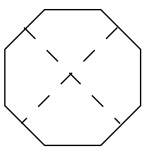
take strip, and fold in half and unfold



place strip on one diagonal fold line and fold down corners



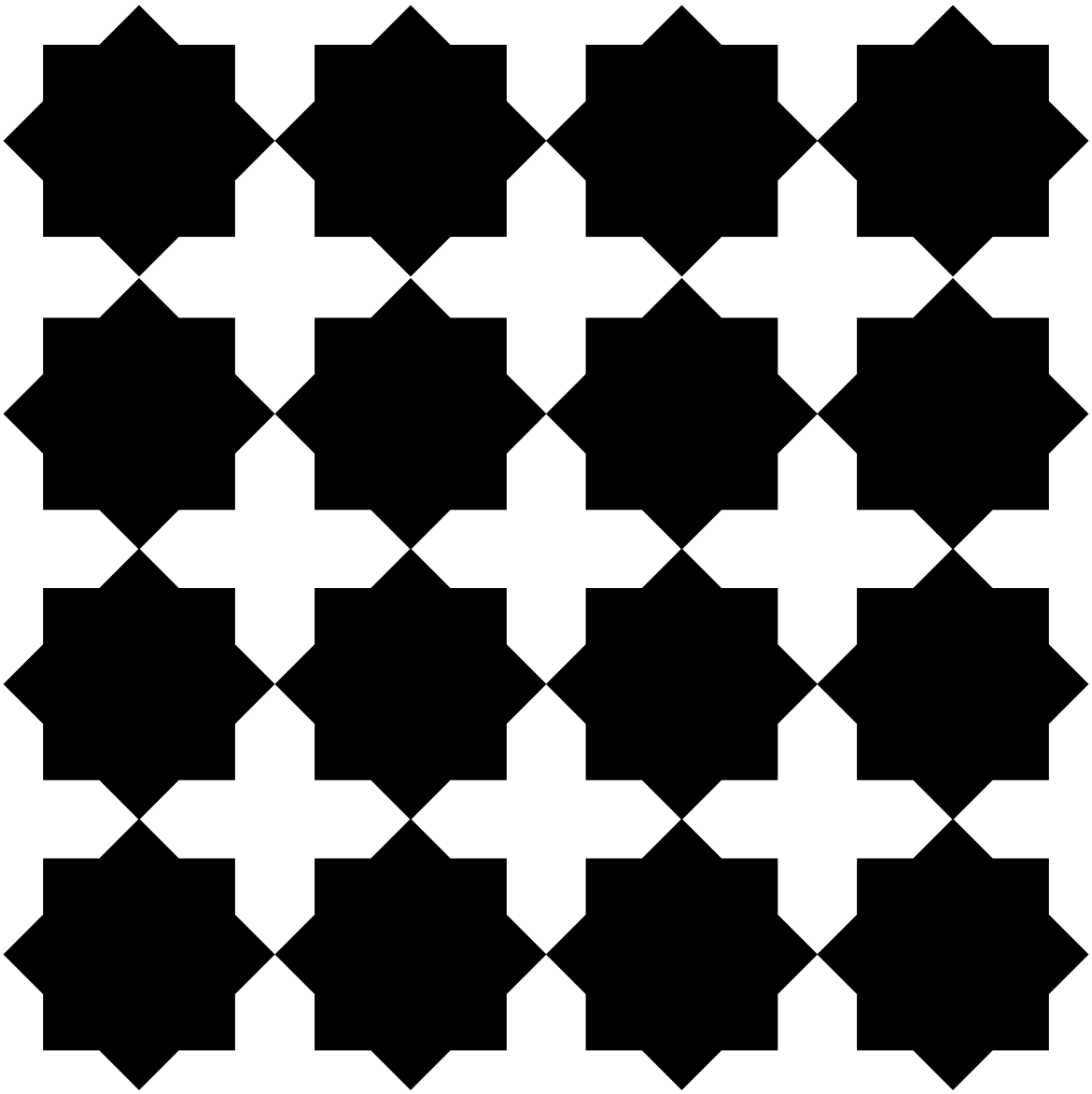
repeat with other diagonal fold line



remove strip and turn over

5.4 Making patterns: Islamic tiling patterns using folded paper shapes

Islamic Tiling Pattern #1 using folded 'A' size squares



Instructions for making the pattern above:

- Make sets of 8-pointed star tiles using 2 folded squares (of the same size and colour) for each one.

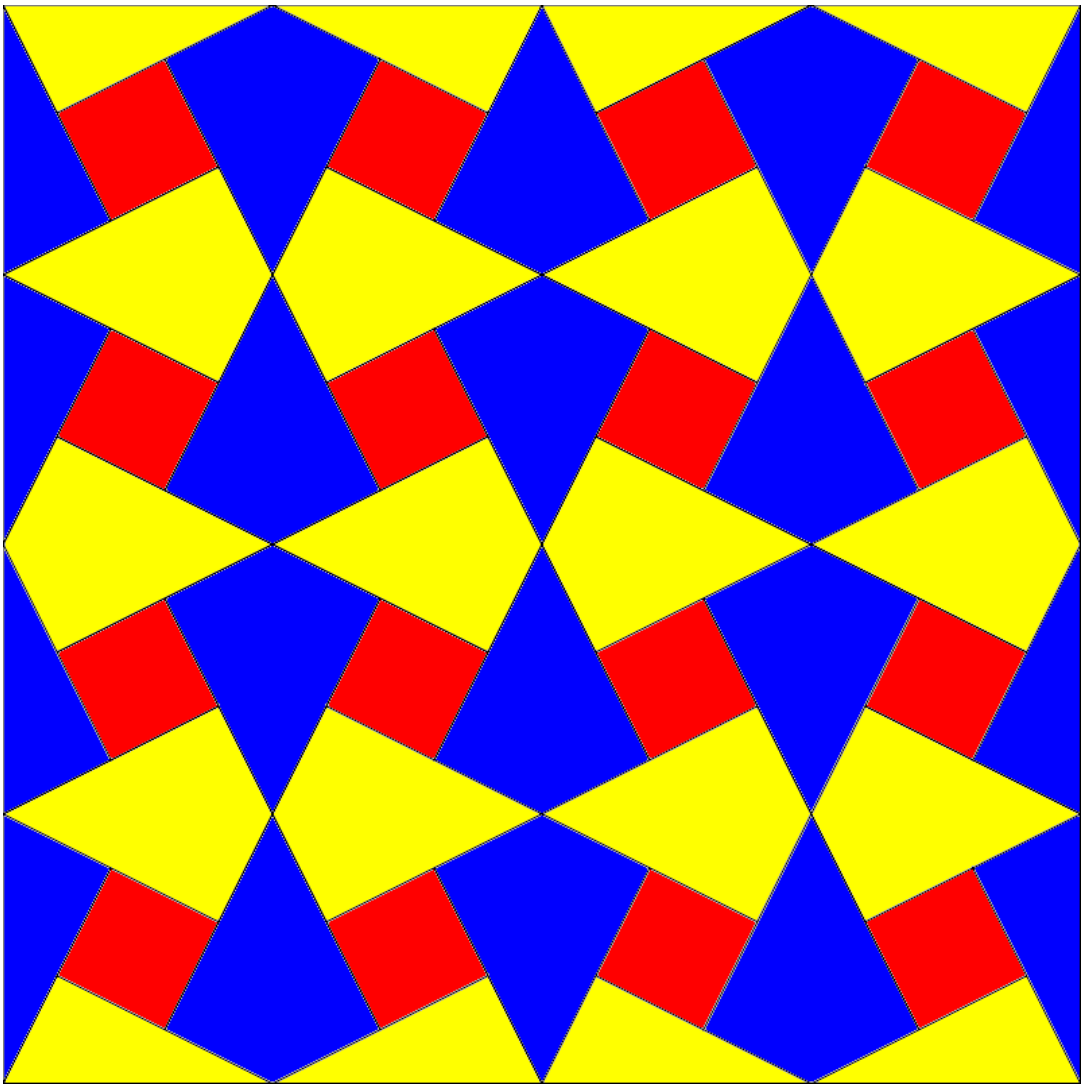
- Place the 8-pointed star tiles onto a large sheet of paper or card horizontally and vertically as in the pattern above.

- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

- The gaps will leave the 'cross' type tile.

Islamic Tiling Patterns #2 using folded 'A' size kites

This is a common Islamic pattern. It is derived from a visual proof of Pythagoras' Theorem by Abu'l Wefa.



Instructions for making the pattern above:

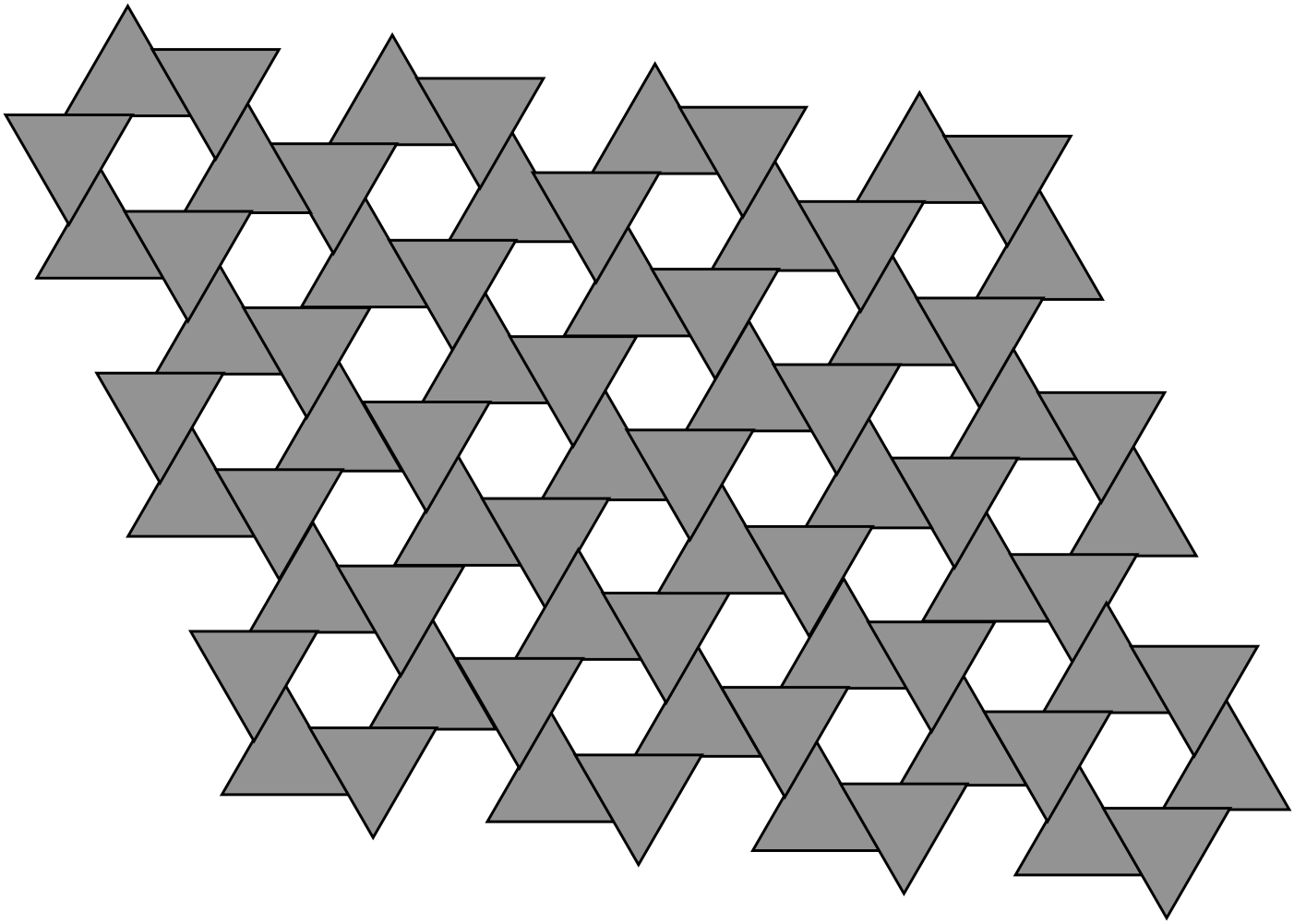
- Make 2 sets of folded kite tiles using two different colours of A size paper.

- Place kite tiles onto a large sheet of paper or card as in the pattern above. Possibly start by placing 4 kites (2 of each colour) around a point and then place other kites to make your pattern grow.

- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

- The gaps will be square.

Islamic Tiling Patterns #3 using folded 'A' size equilateral triangles



Instructions for making the pattern above:

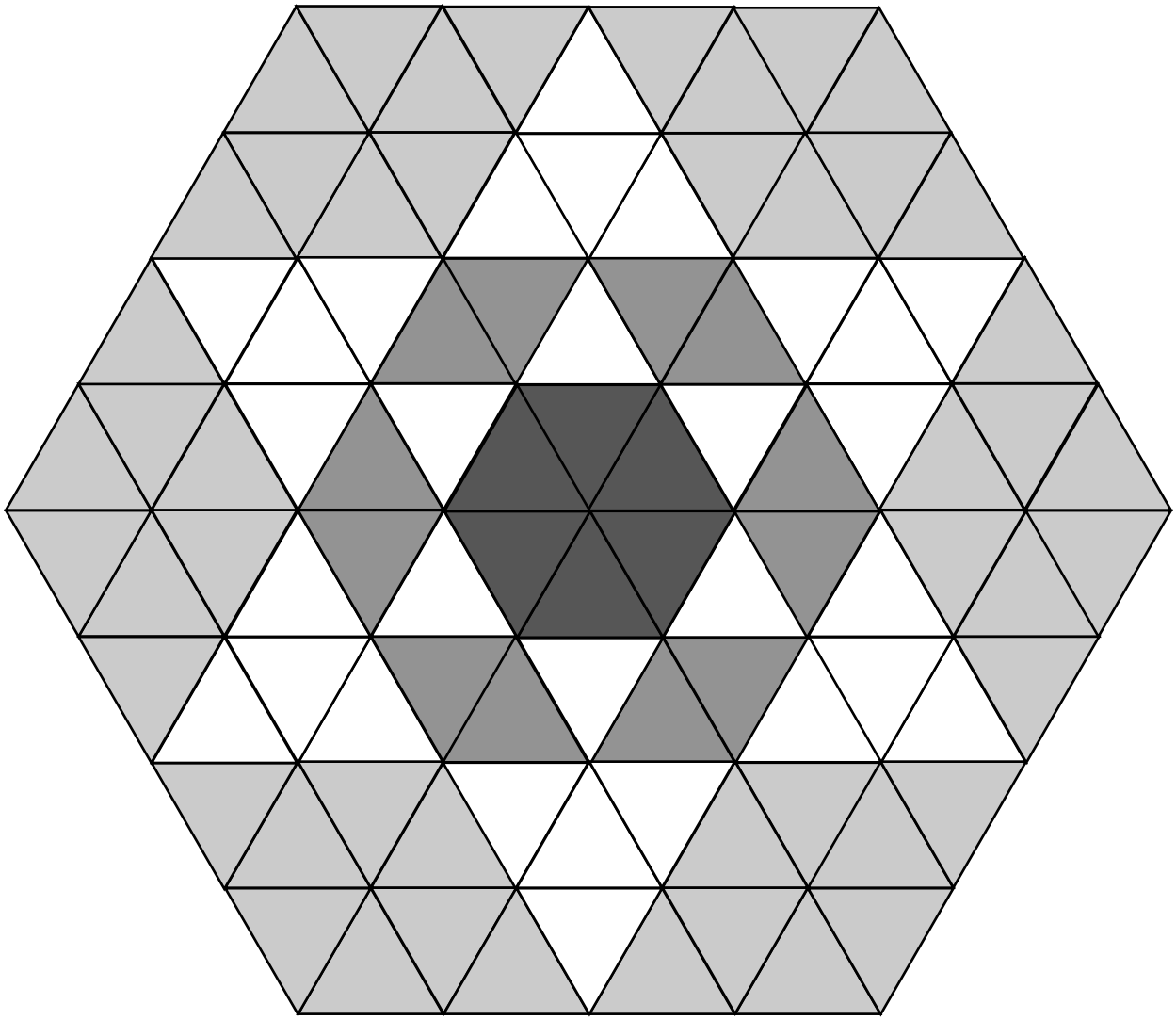
- Make a number of folded equilateral tiles using coloured A size paper.

- Place the equilateral tiles onto a large sheet of paper or card as in the pattern above. Possibly start by placing 6 triangles with overlapping edges around in a loop so that a hexagon is formed in the gap in the middle. Then place other triangles to make your pattern grow like the one above.

- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

- The gaps will be hexagons.

Islamic Tiling Patterns #4 using folded 'A' size equilateral triangles



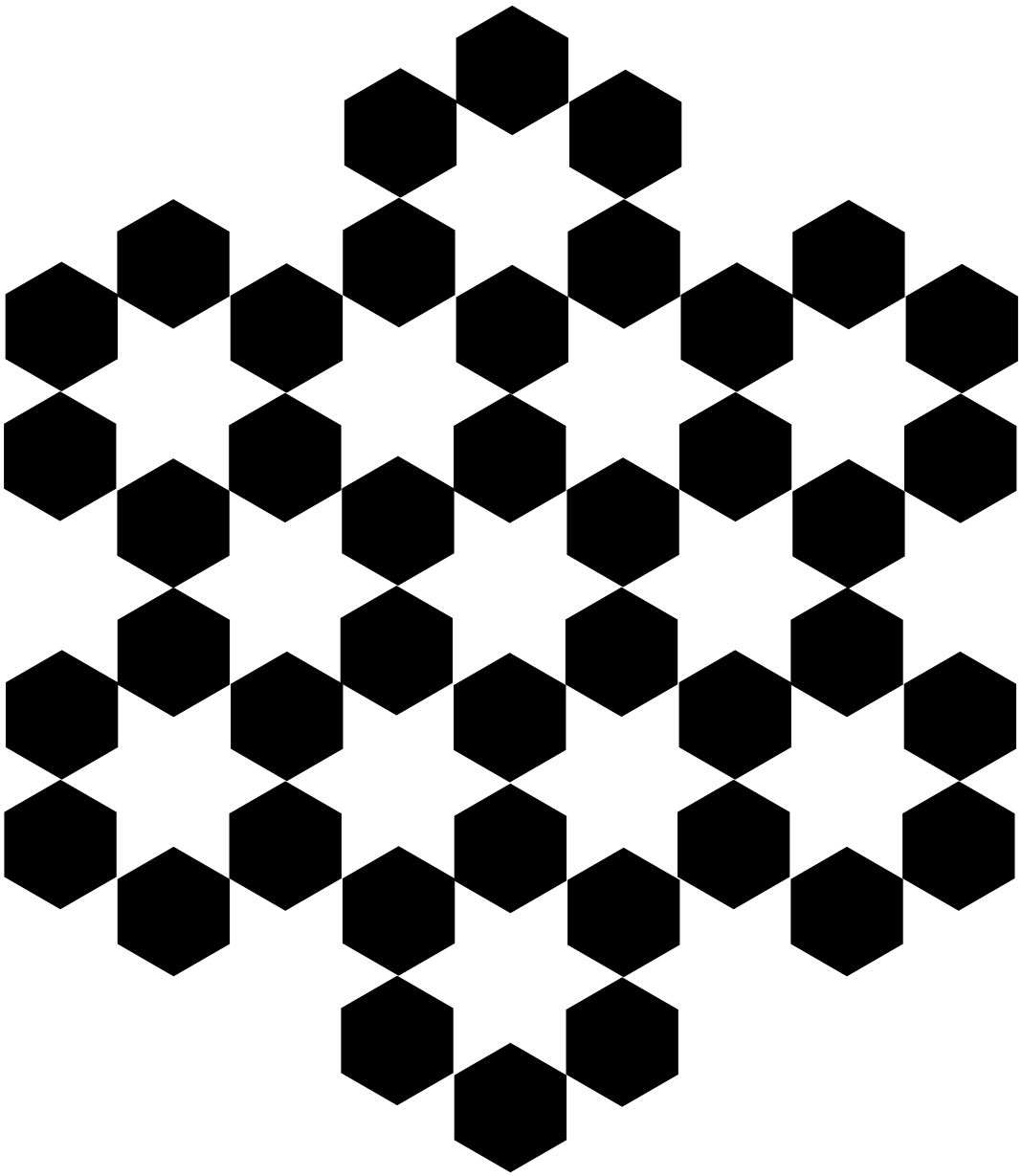
Instructions for making the pattern above:

- Make 2 sets of folded equilateral tiles using different colours of the same A size paper.

- Place the equilateral tiles onto a large sheet of paper or card as in the pattern above. Possibly start by placing 6 triangles together to make a hexagon. Then place 6 different triangles on the edges of the hexagon to make a 6-pointed star. Then place 12 triangles like the ones you started with to make your 6-pointed star into a larger hexagon. Continue placing triangles to make your pattern grow like the one above.

- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

Islamic Tiling Patterns #5 using folded 'A' size equilateral triangles



Instructions for making the pattern above:

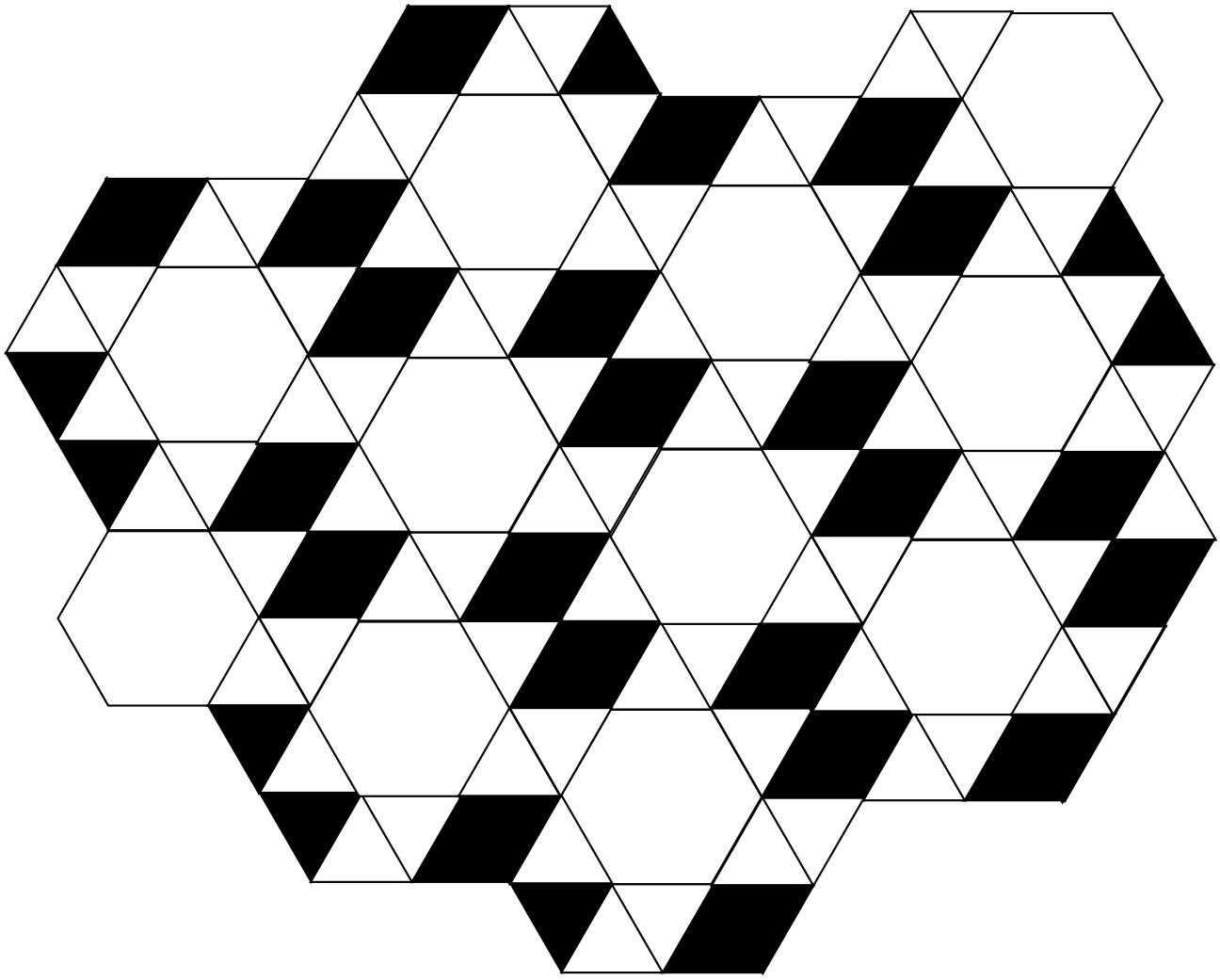
- Make sets of 6-pointed stars using pairs of folded equilateral tiles made from coloured A size paper.

- Place the 6-pointed star tiles onto a large sheet of paper or card horizontally and vertically as in the pattern above.

- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

- The gaps should be hexagons.

Islamic Tiling Patterns #6 using folded 'A' size equilateral triangles



Instructions for making the pattern above:

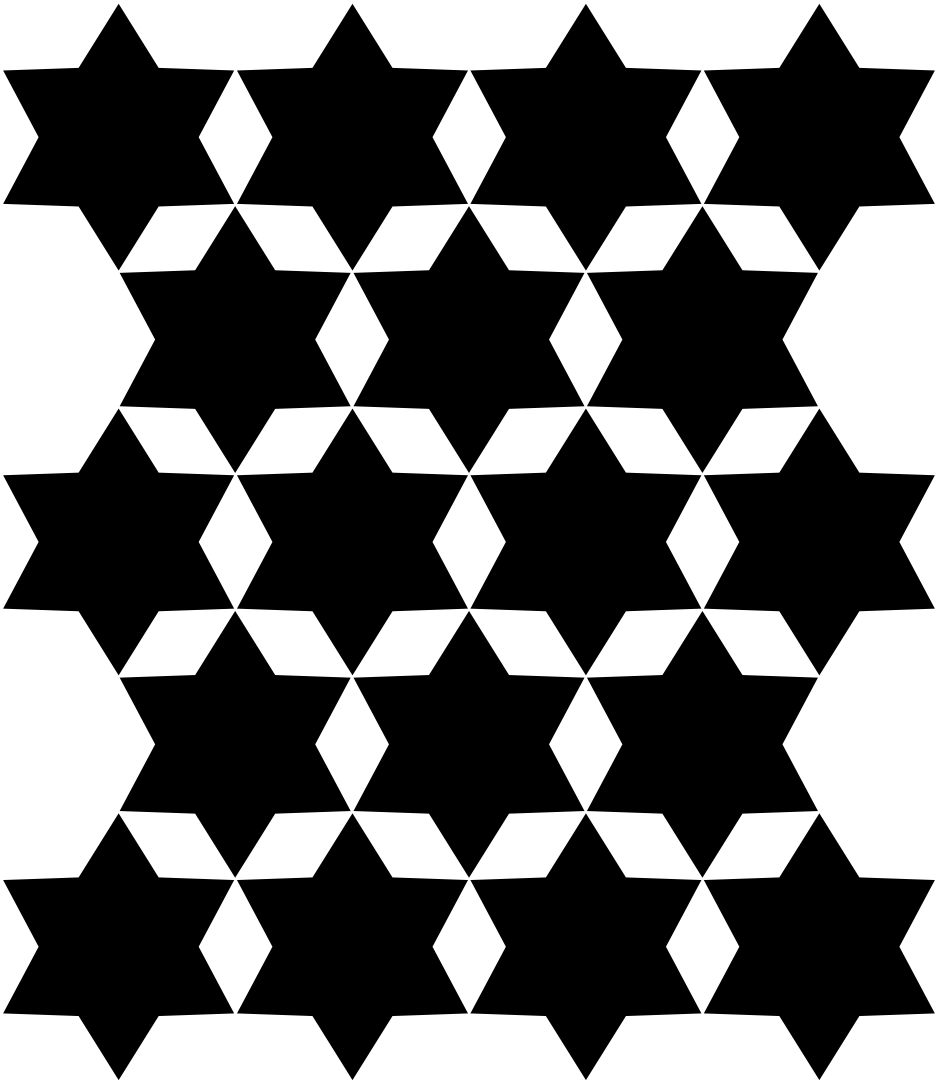
- Make sets of 6-pointed stars using pairs of folded equilateral tiles made from coloured A size paper.

- Place the 6-pointed star tiles onto a large sheet of paper or card diagonally one edge-to-edge as in the pattern above.

- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

- The gaps should be pairs of rhombi.

Islamic Tiling Patterns #7 using folded 'A' size equilateral triangles



Instructions for making the pattern above:

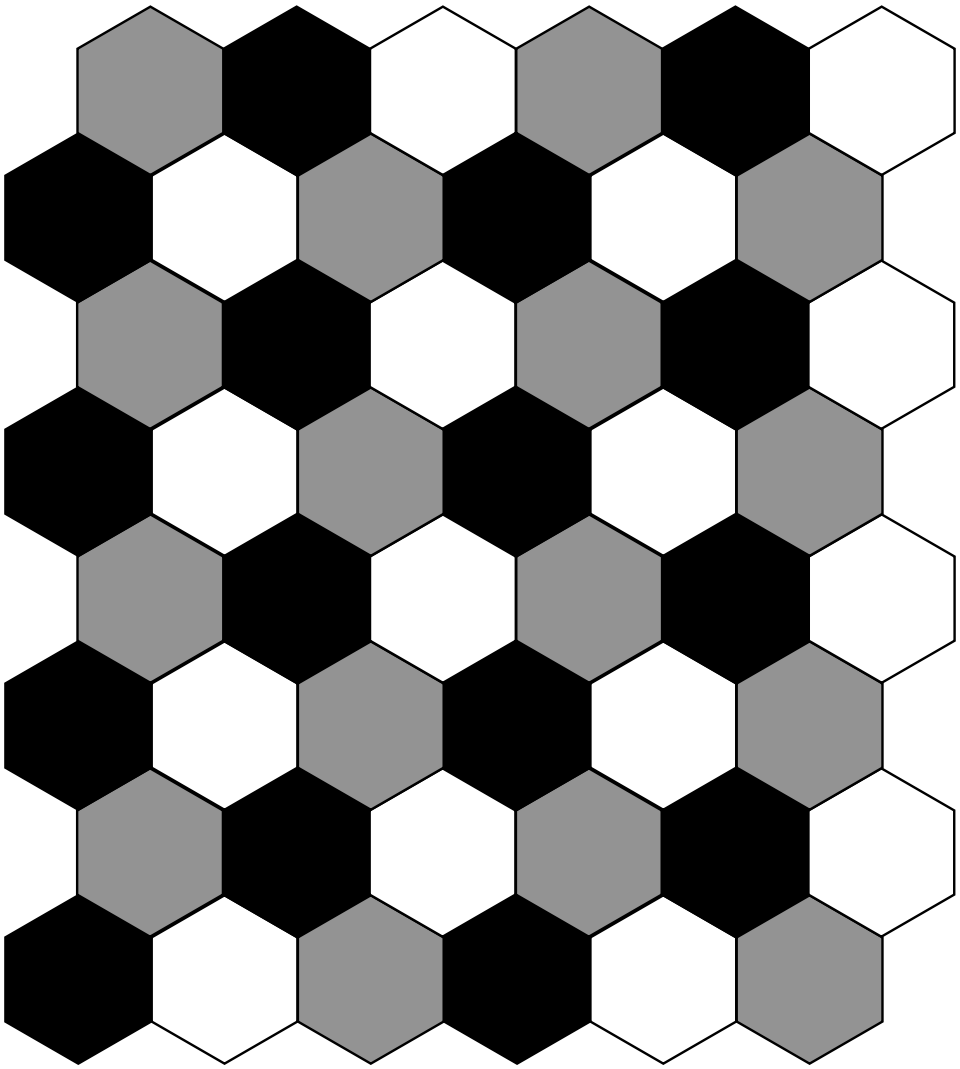
- Make sets of 6-pointed stars using pairs of folded equilateral tiles made from coloured A size paper.

- Place the 6-pointed star tiles onto a large sheet of paper or card horizontally and vertically as in the pattern above.

- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

- The gaps should consist of 3 rhombi at a point.

Islamic Tiling Patterns #8 using folded 'A' size hexagons



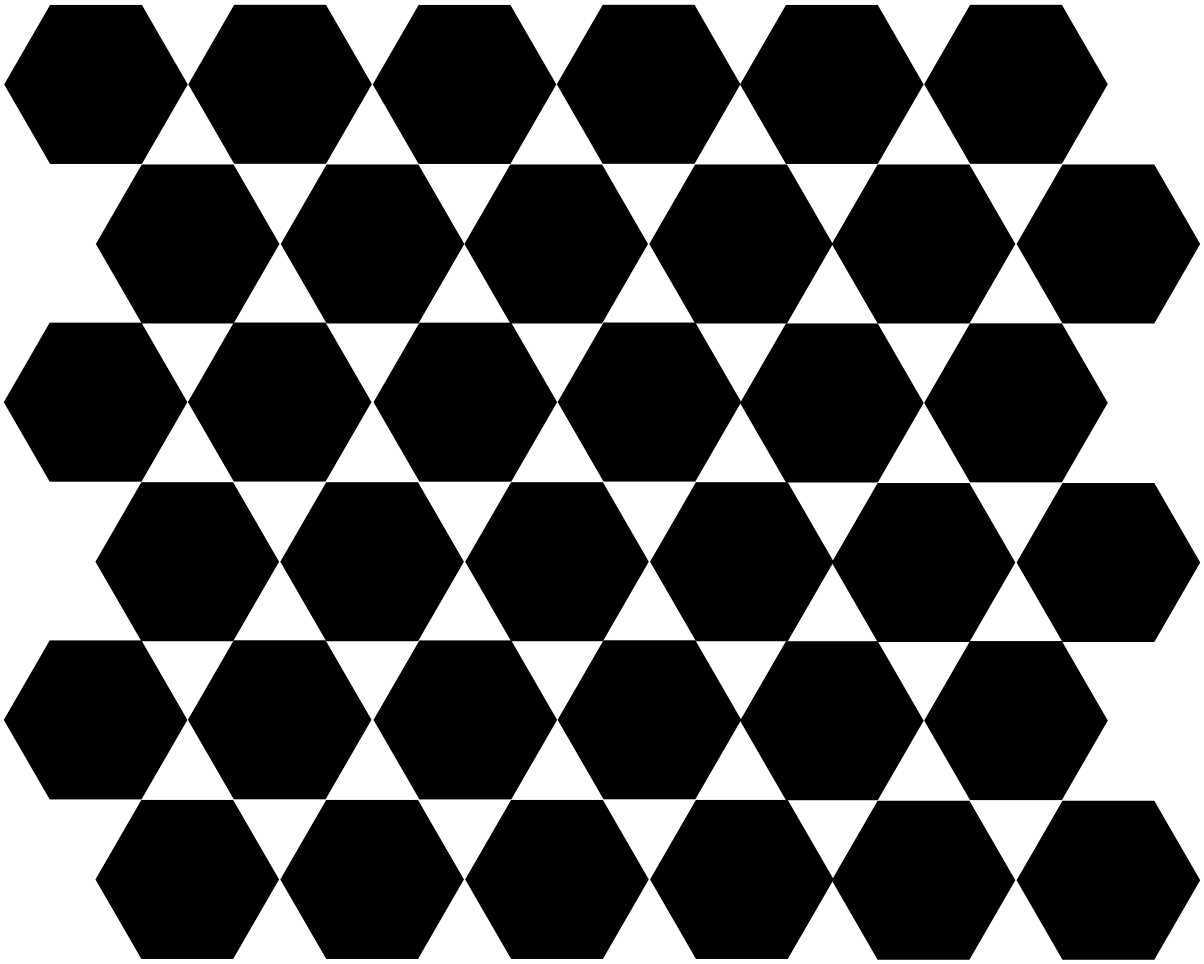
Instructions for making the pattern above:

• Make 3 sets of folded hexagon tiles made from different coloured A size paper.

• Place the hexagon tiles onto a large sheet of paper or card as in the pattern above.

• When satisfied with your positioning, glue the tiles onto the sheet of paper or card.

Islamic Tiling Patterns #9 using folded 'A' size hexagons



Instructions for making the pattern above:

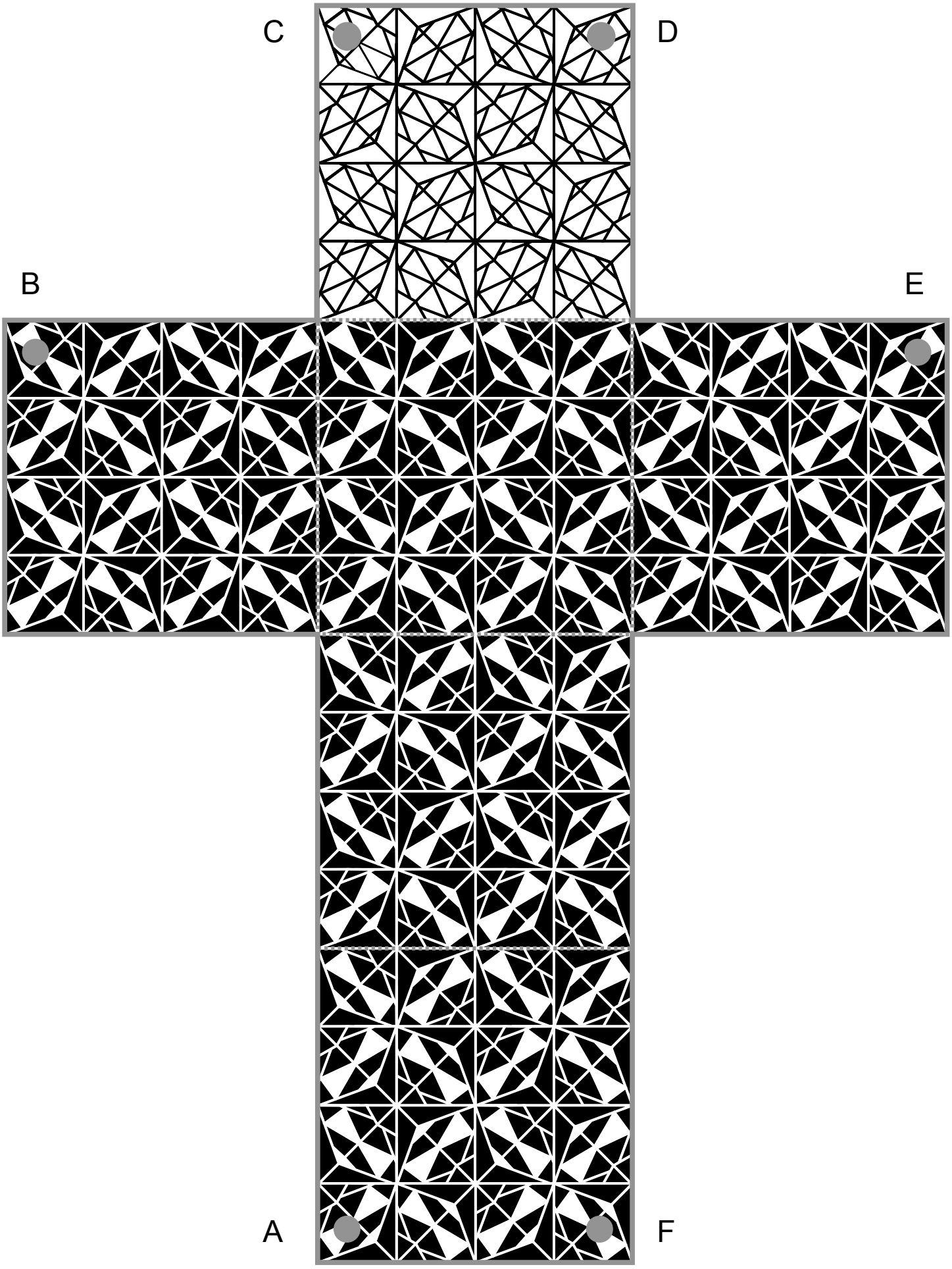
- Make a number of folded hexagon tiles made from coloured A size paper.
- Place the hexagon tiles onto a large sheet of paper or card horizontally and diagonally as in the pattern above.
- When satisfied with your positioning, glue the tiles onto the sheet of paper or card.
- The gaps should be equilateral triangles.
- Look at the pattern - you might see it as hexagons surrounded by triangles or 6-pointed stars with hexagons in the middle.

5.5 Making polyhedra: patterned Platonic polyhedra using pull-up nets

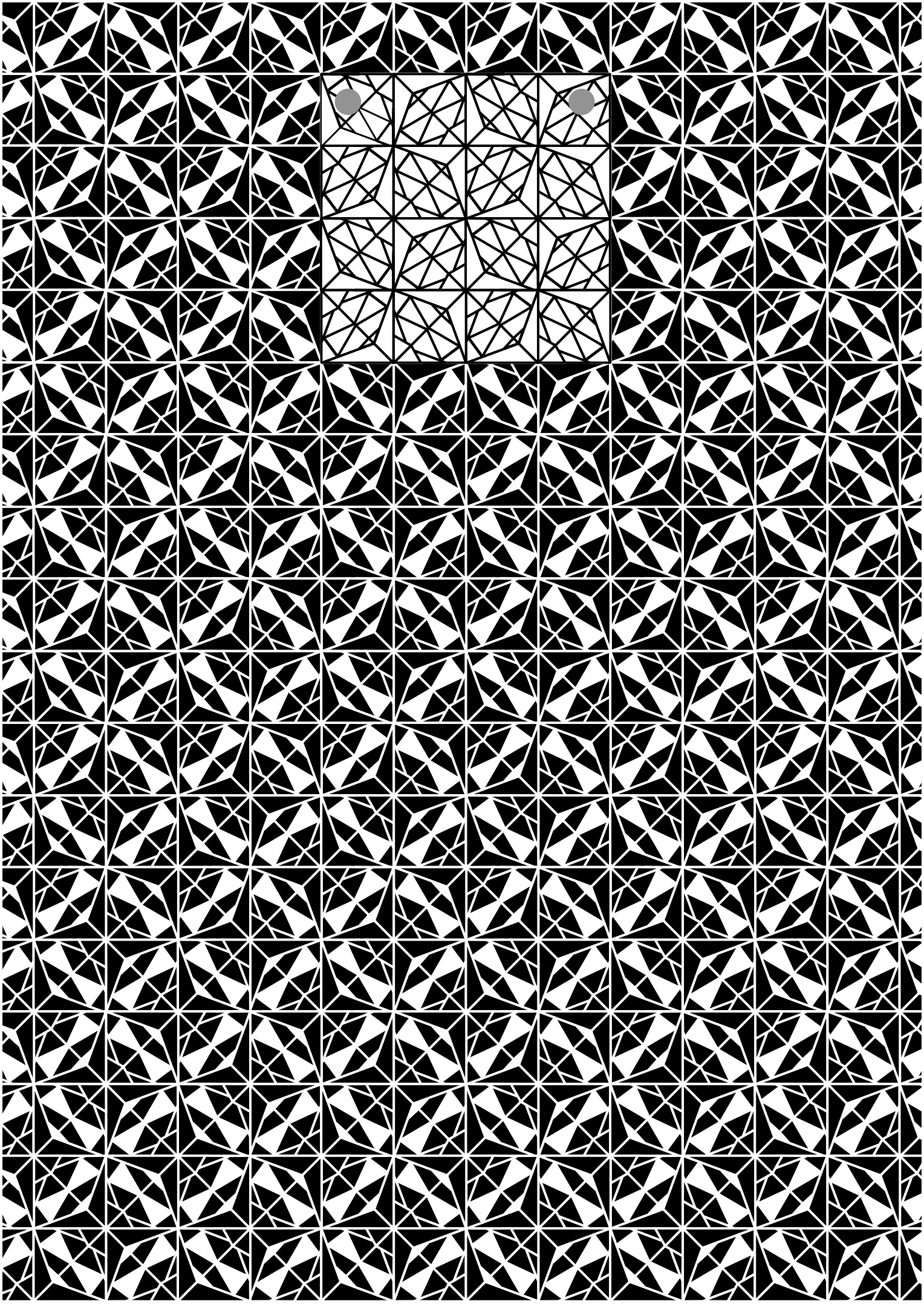
- Use the prepared nets and associated backing sheets to create polyhedra.
- When you think you've got the hang of it, try creating your own nets for pull-up polyhedra.
- Are there other ways you could make a cube, or some of the other polyhedra?

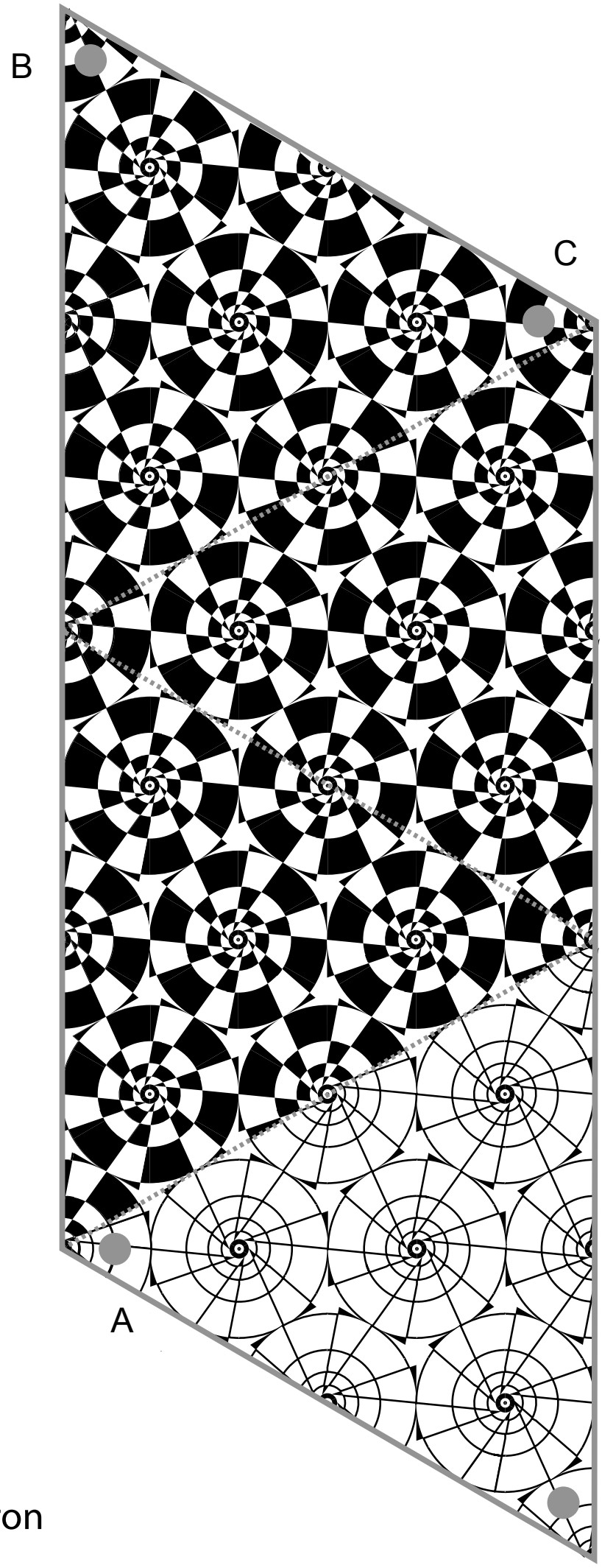
Basic instructions:

1. Cut out the net for your pull-up polyhedron.
2. Use a ruler and sharp point to score lightly along the faint grey dotted lines.
3. Make holes at the points A, B, C, D, etc.
4. Stick the light coloured polygon of the net onto the associated light coloured polygon on the A4 backing card with the **same pattern**.
5. Punch the two remaining holes in the backing card.
6. Thread and weave thin string through the holes A, B, C, D, etc, in that order to link the polygons together.
7. Gently pull up the net to make your polyhedron sit on the backing card.

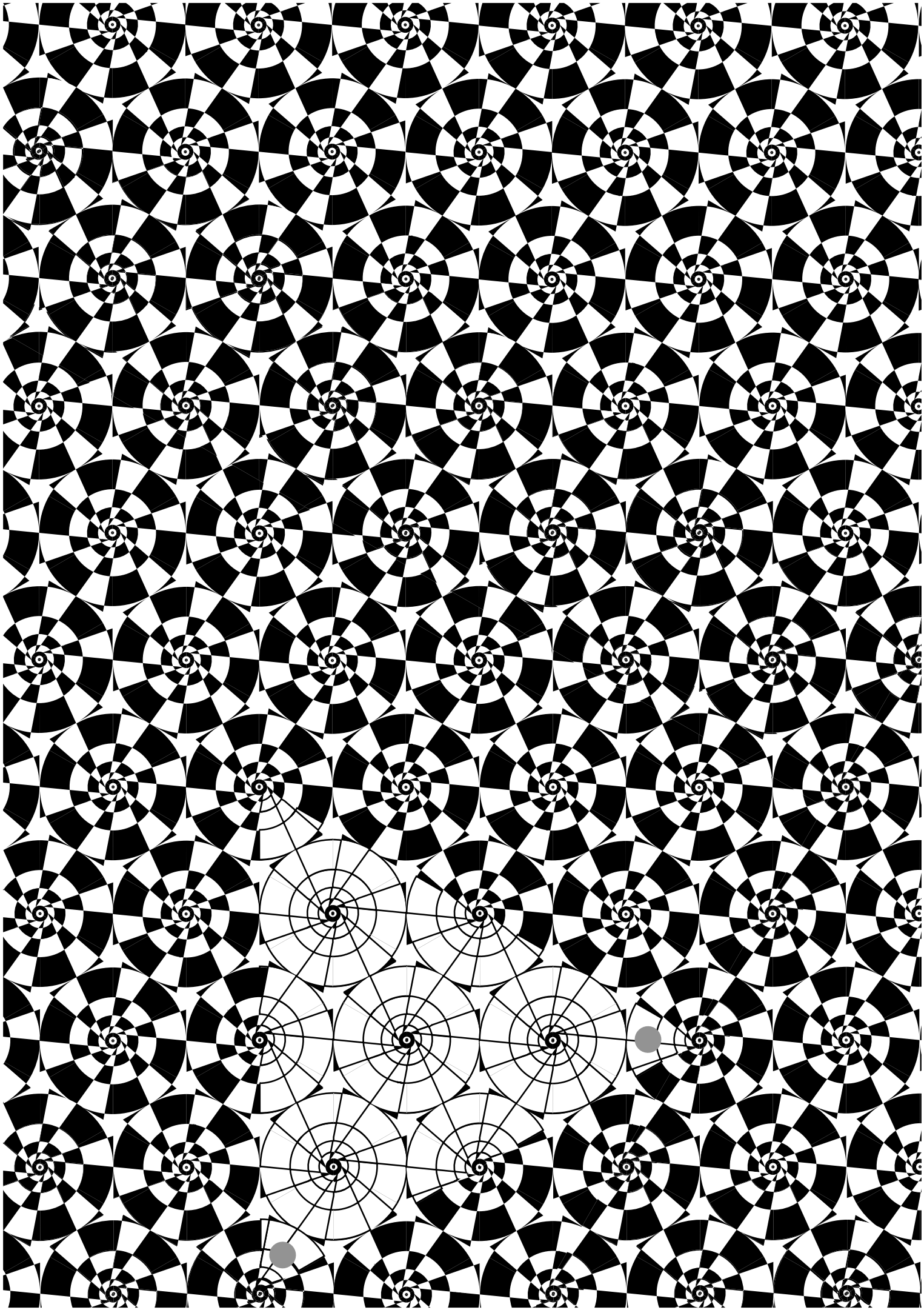


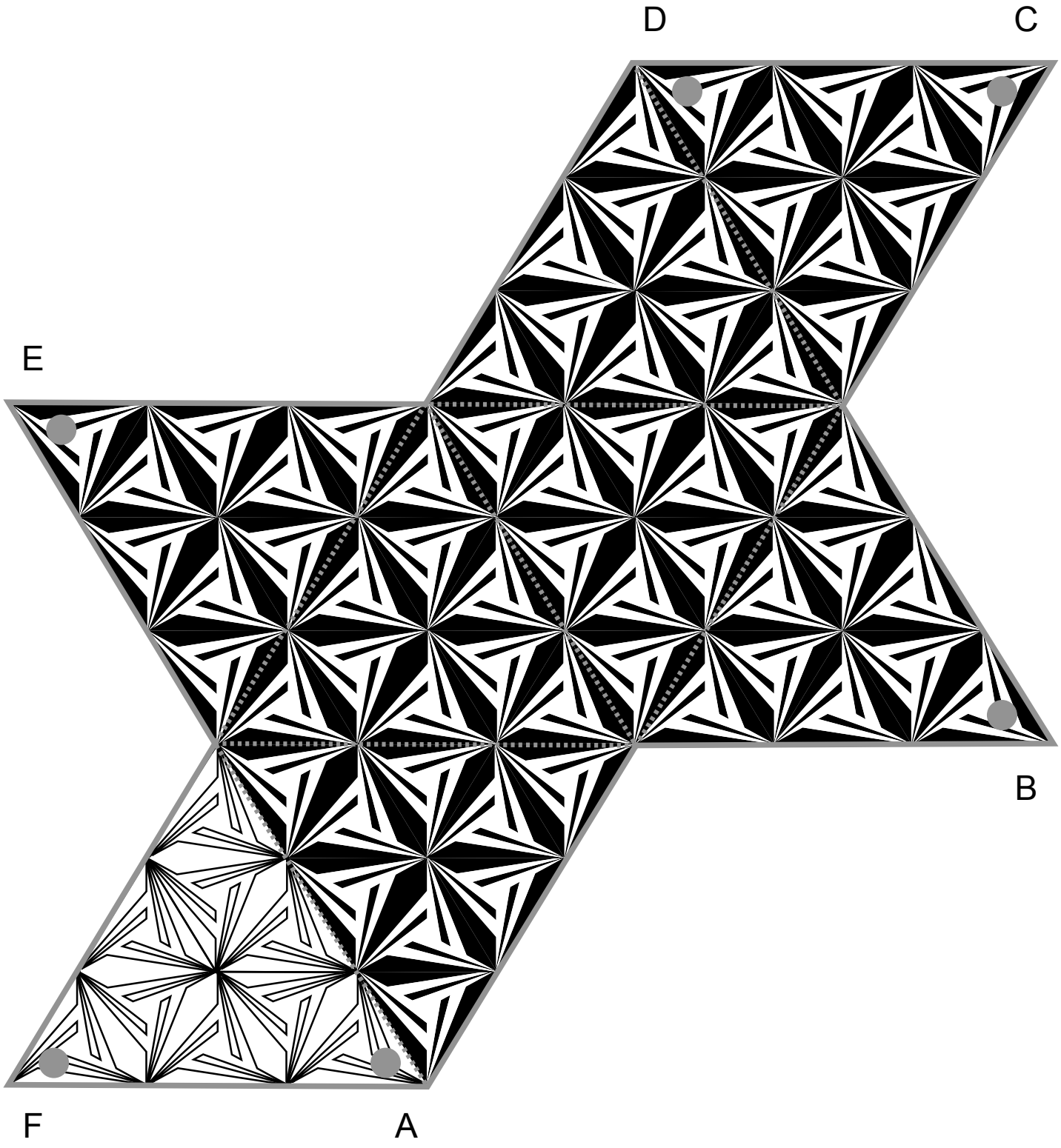
Cube



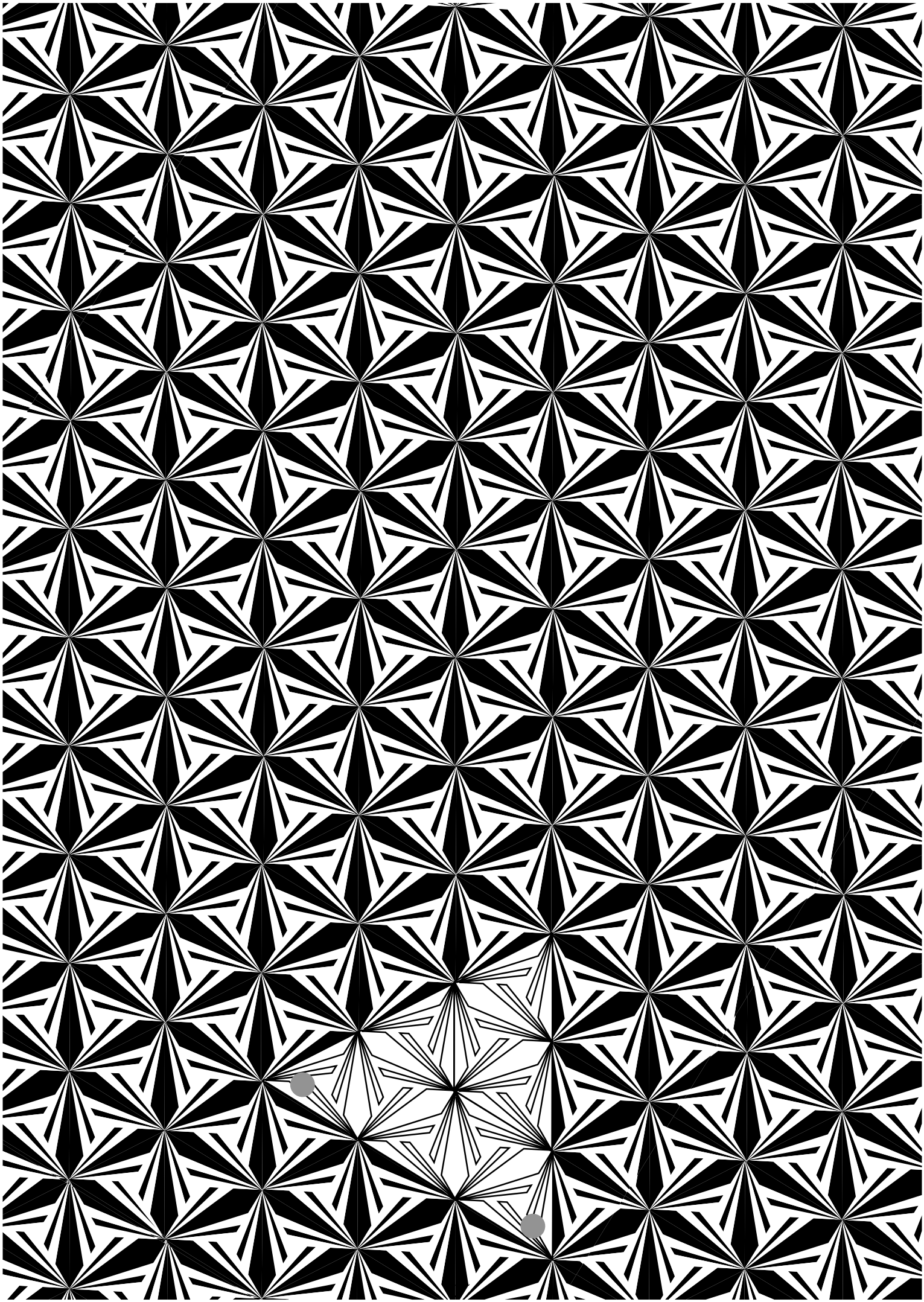


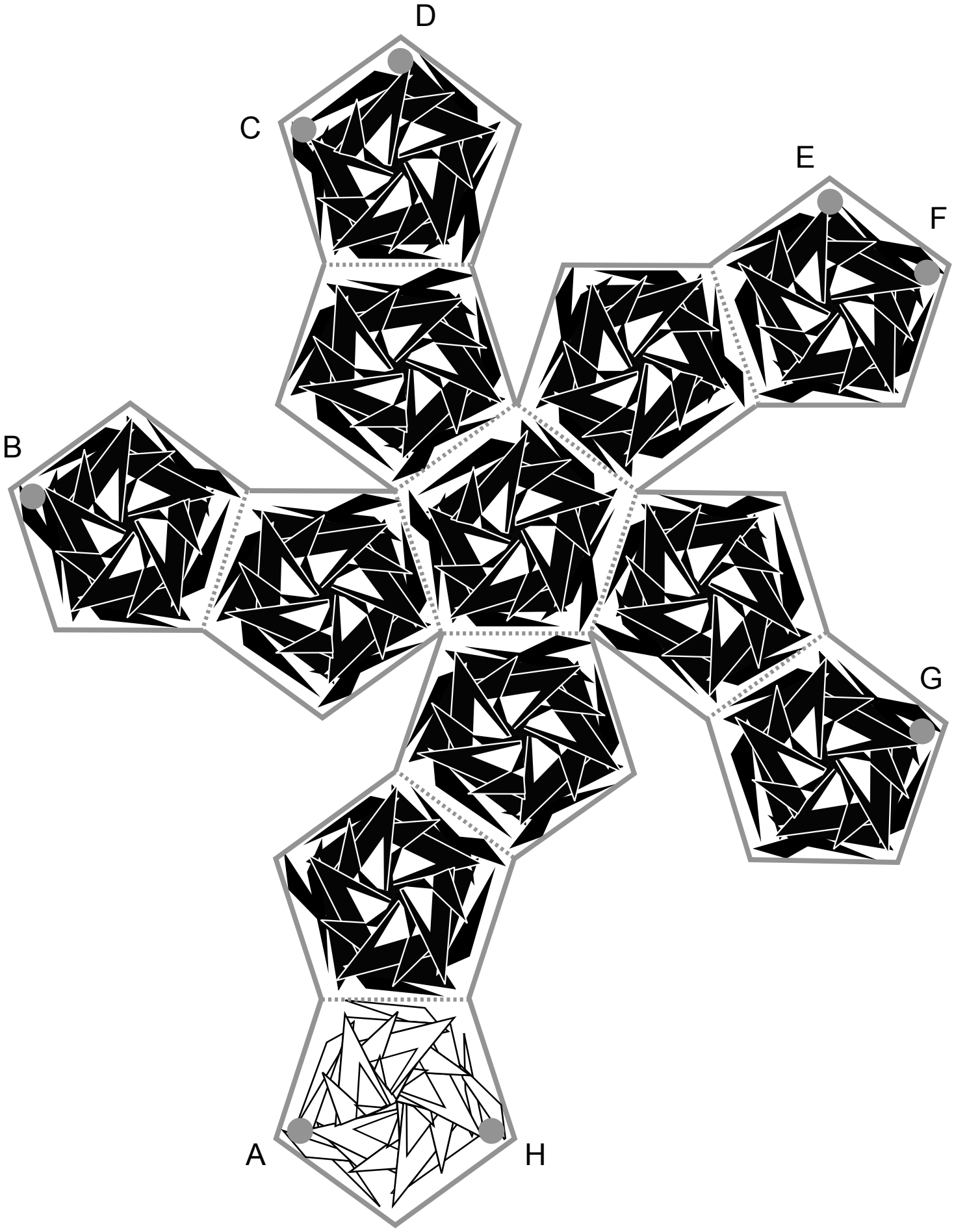
Tetrahedron



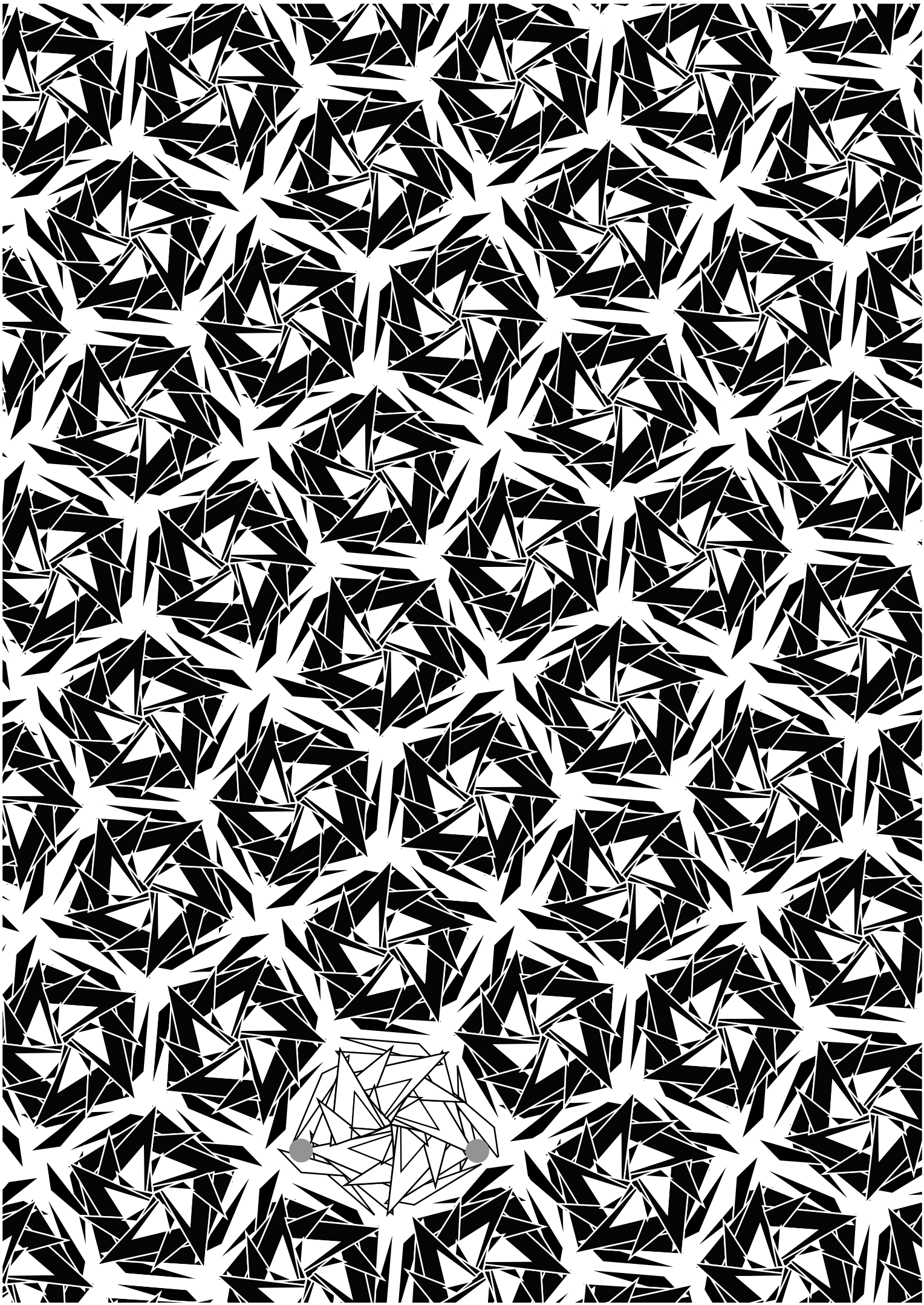


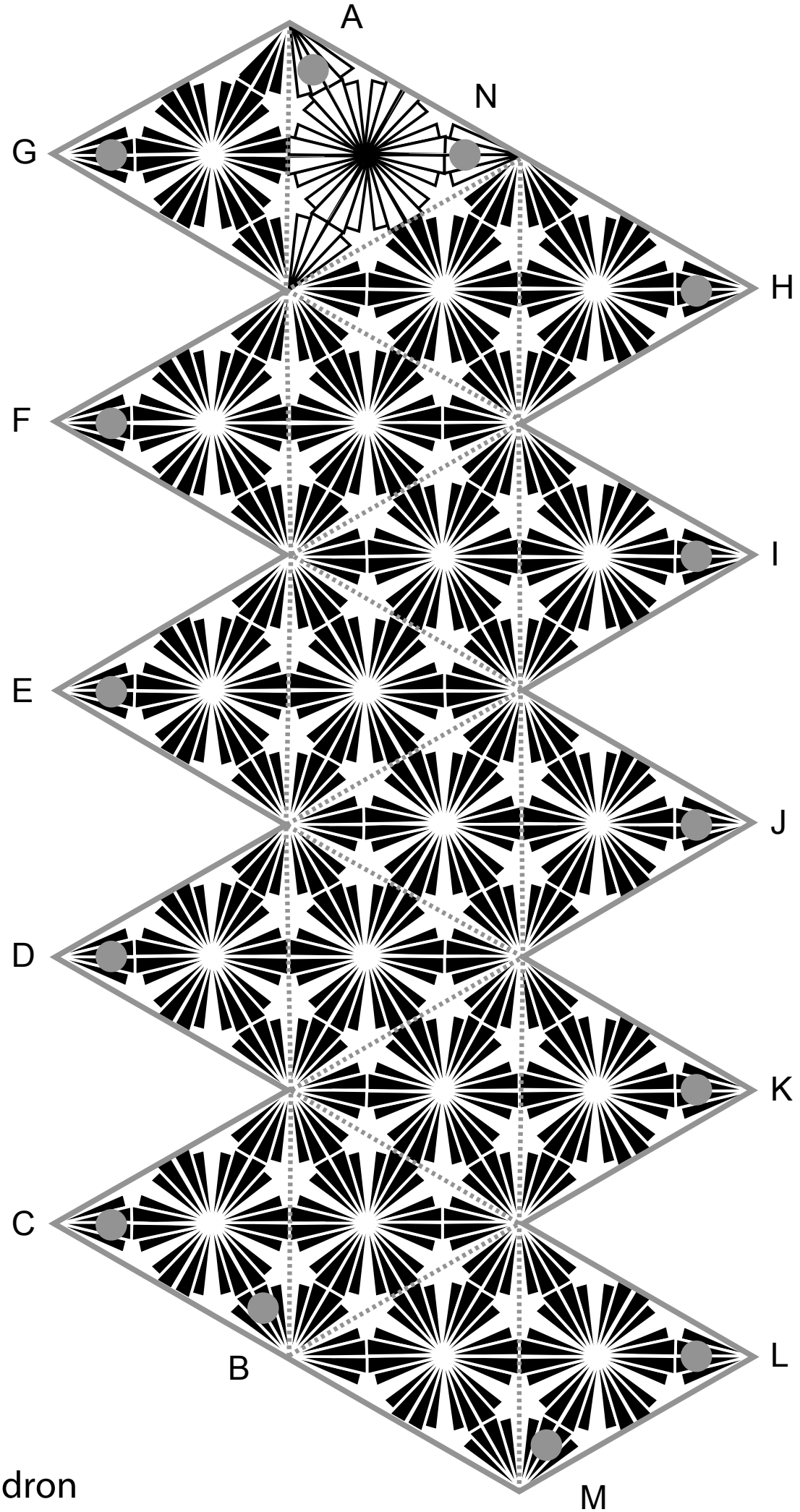
Octahedron



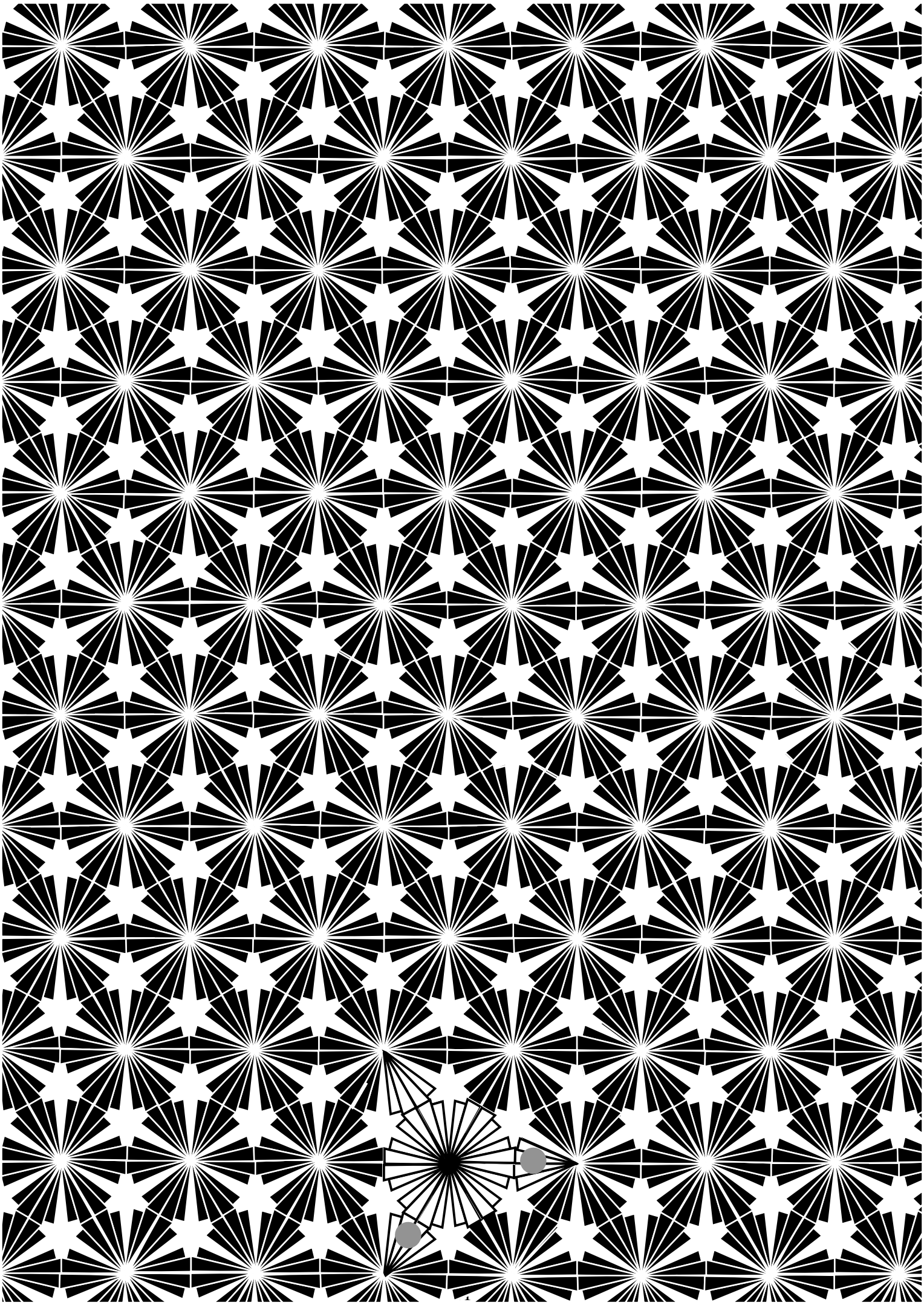


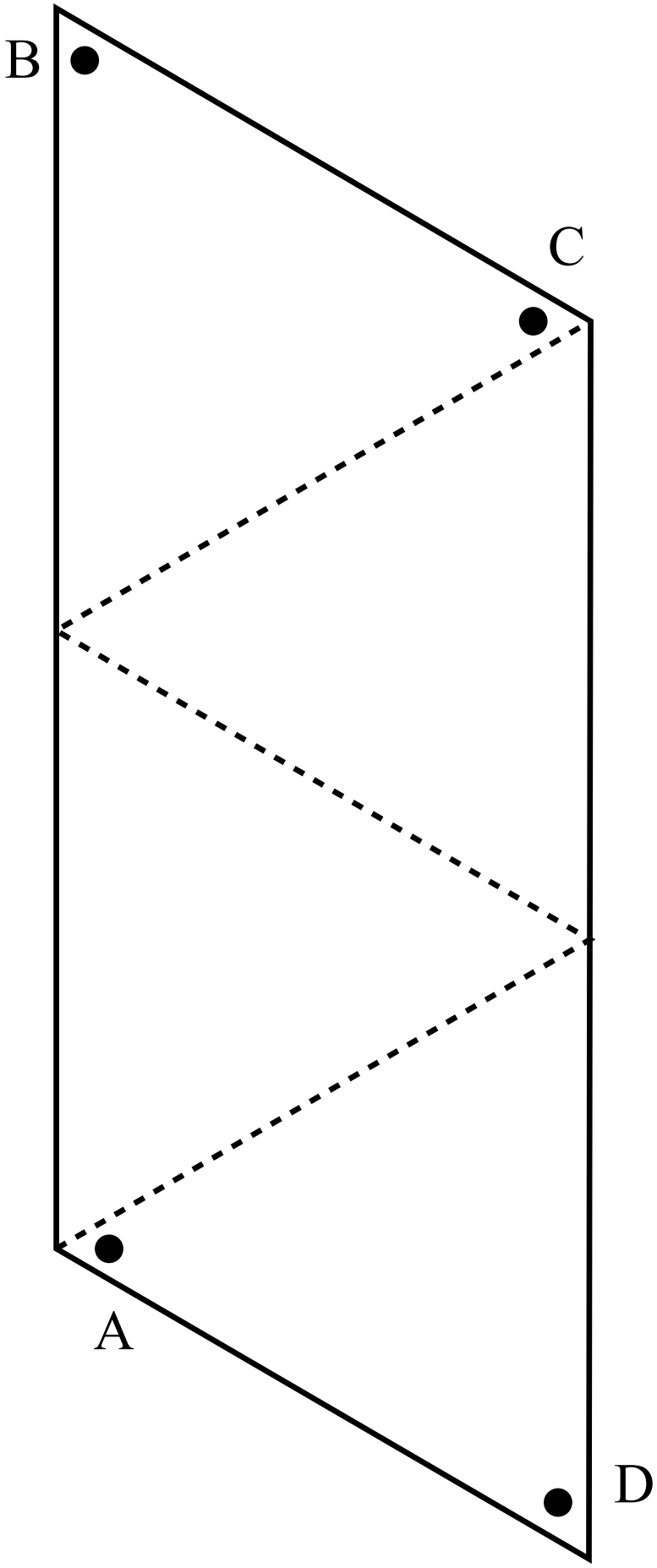
Dodecahedron

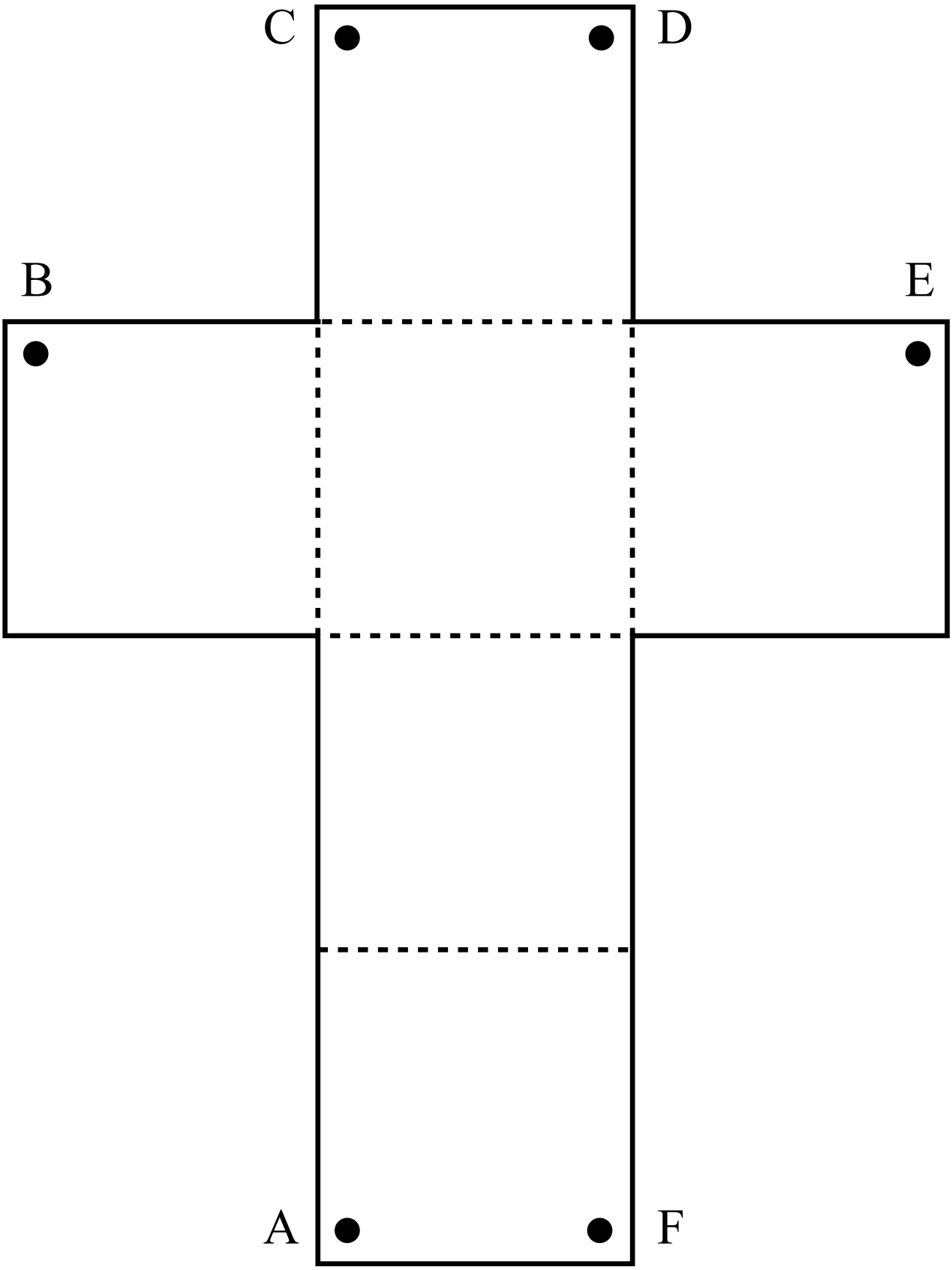




Icosahedron




B
C
A
D


C
D
B
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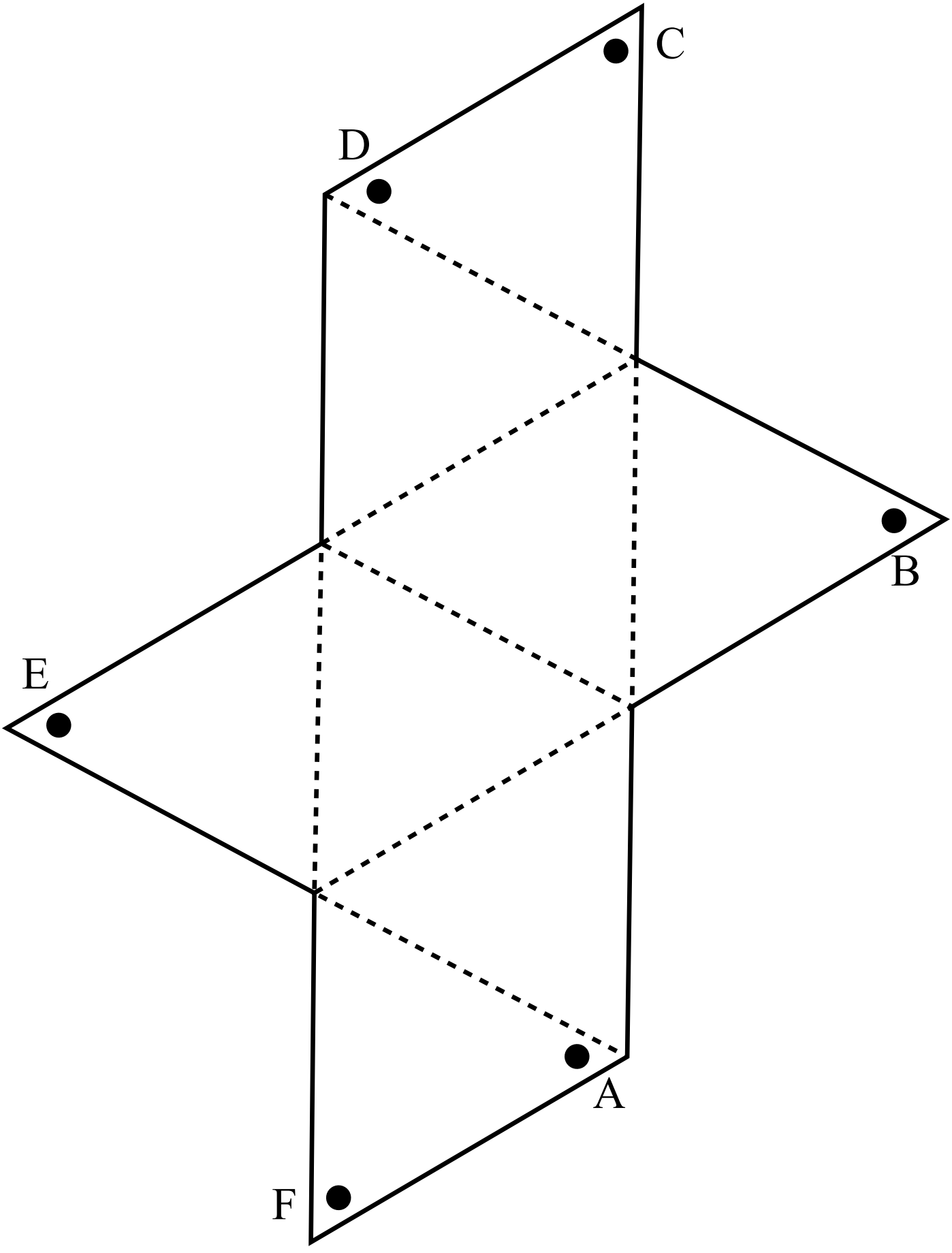
A
F


C
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B
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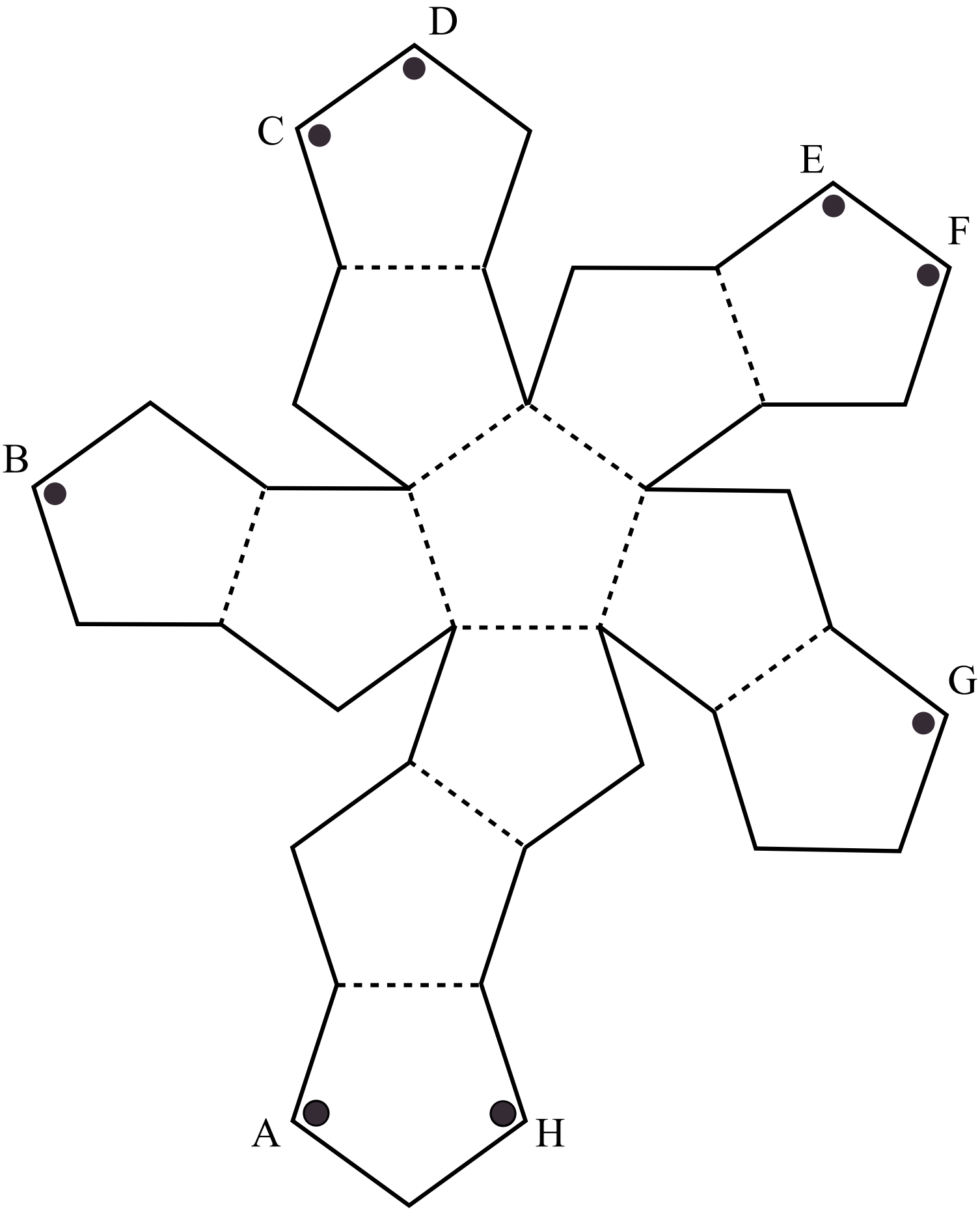
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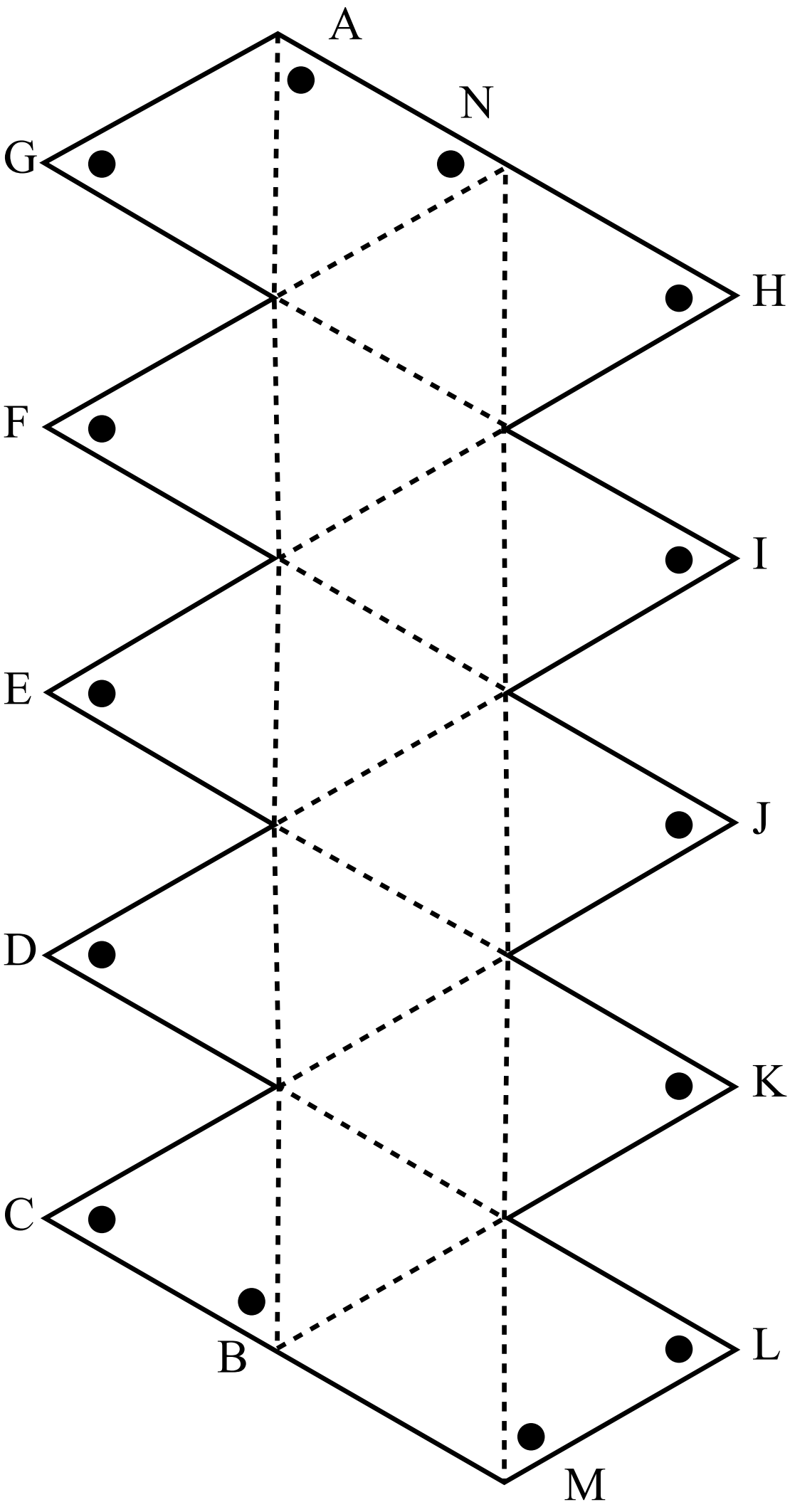
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